UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2700 — Introduction to Mathematical Finance and Investment Theory

Day of examination: Friday, December 13th, 2013

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider a single-period market consisting of a bank account $B = \{B_0, B_1\}$ and one risky asset $S = \{S_0, S_1\}$. The current values of the assets are $B_0 = 1$ and $S_0 > 0$. The risk-free interest rate is $r > 0$. The sample space Ω consists of two states ω_1 and ω_2, where the probability of ω_1 occurring is $p \in (0, 1)$. The risky asset can either go up to $S_1 = uS_0$ (if $\omega = \omega_1$) or down to $S_1 = dS_0$ (if $\omega = \omega_2$), where $u > 1$ and $d < 1$ are given numbers.

1a

Consider a claim X_u that gives 1 krone in the up state and nothing in the down state. Determine the trading strategy that generates X_u. What is the initial portfolio value?

Similarly, determine the trading strategy and initial portfolio value of the claim X_d that gives 1 krone in the down state and nothing in the up state.

1b

Show that the risk-neutral probability measure is $Q = \left(\frac{1+r-d}{u-d}, \frac{u-(1+r)}{u-d}\right)^T$, provided $u > 1 + r$.

1c

Take $S_0 = 100$, $u = 11/10$, $d = 9/10$. State the condition on r ensuring that the market is complete, and then use the risk-neutral valuation formula to find the prices of call and put options with exercise prices respectively 109 (call) and 91 (put).

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1d

Consider the general market model given at the beginning of Problem 1. Denote by $U(w)$ the utility function

$$U(w) = \ln(w), \qquad w > 0.$$

Use the (two-steps) risk-neutral computational approach to solve the problem of maximizing expected utility of terminal wealth, with initial wealth $V_0 = \nu$, for a given $\nu > 0$:

$$\max_{H \in \mathbf{R}^2} E\left[U\left(V_1\right)\right], \qquad V_0 = \nu,$$

where $V_1 = H_0 B_1 + H_1 S_1$ is the terminal wealth. Justify your arguments (derive the formulas you use).

Problem 2

A stock price is currently \$100. In three months it will be either \$125 (with probability $2/5$) or \$80 (with probability $3/5$). If it rises to \$125 in three months, then it will either be \$150 (with probability $2/5$) or \$100 (with probability $3/5$) after another three months. If it drops to \$80 after the first three months, then after another three months it will be either \$100 (with probability $2/5$) or \$75 (with probability $3/5$). The bank process is $B_0 = 1$, $B_1 = 1 + r$, and $B_2 = (1 + r)^2$, where $r > 0$ is the per annum interest rate.

2a

Identify the stock prices S_t at the trading dates $t = 0, 1, 2$, and the probability $P(\omega)$ attached to each "state of the world" ω. Determine the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,1,2}$ generated by the price process $S = \{S_0, S_1, S_2\}$ (justify your answer).

2b

Show that the risk-neutral probability (martingale) measure

$$Q = (Q_1, Q_2, Q_3, Q_4)^T$$

is given by

$$Q_1 = \frac{2}{9}(1 + 5r)^2, \quad Q_2 = \frac{2}{9}\left(1 - 25r^2\right),$$
$$Q_3 = \frac{1}{9}(1 - 4r)(16r + 1), \quad Q_4 = \frac{4}{9}(1 - 4r)^2,$$

provided $r < 1/5$.

2c

Let X be the payoff of a put option that expires at $T = 2$ with an exercise price $K = 75 + (1 + r)^2$. Take $r = 1/10$ and use the risk-neutral valuation formula to compute the arbitrage-free price of the claim X at time $t = 0$.

Determine the replicating portfolio for this put option, i.e., the trading strategy that generates X. Work backwards in time, deriving simultaneously the value process V and trading strategy H.

(Continued on page 3.)

Problem 3

Consider the single-period market model from Problem 1. Given a trading strategy $H = (H_0, H_1)^T$, denote by $V = \{V_0, V_1\}$ the portfolio value. The returns of the (bank/stock) assets are denoted by R_0, R_1, while R denotes the portfolio return.

3a

Show that the portfolio value at time $t = 1$ can be expressed as

$$V_1 = V_0\big(1 + r + F(R_1 - r)\big), \qquad F = \frac{H_1 S_0}{V_0},$$

and then compute R_1, V_1 using the values of S in Problem 1.

3b

An investor is prohibited by stock exchange rules from selling stocks short. This imposes a constraint on the admissible trading strategies. Denote the set of admissible strategies by $\mathbb{K}$. Express the set $\mathbb{K}$ in terms of the fraction F introduced in (3a). Compute the support function $\delta(\kappa)$ of $-\mathbb{K}$ and its effective domain $\tilde{\mathbb{K}}$.

3c

Make use of the risk-neutral probability approach to solve the constrained maximization problem

$$\max_{F \in \mathbb{K}} E\left[U\big(\nu(1 + r + F(R_1 - r))\big)\right], \qquad V_0 = \nu > 0,$$

where the utility function is $U(w) = \ln(w)$. You can take $p = 1/2$, $r = 1/10$, $u = 12/10$, and $d = 8/10$. [Hint: If necessary, use results from Problem 1 to simplify the computations.]

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