UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3100 — Linear Optimization
Day of examination:	Tuesday, June 12th, 2018
Examination hours:	14.30 – 18.30

This problem set consists of 6 pages.

Appendices:	None
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Consider the LP problem

$$\begin{aligned} &\max\ 5x_1 + 10x_2 \\ &\text{subject to} \\ &x_1 + 3x_2 \leq 50, \\ &4x_1 + 2x_2 \leq 60, \\ &x_1 \leq 5, \\ &x_1, x_2 \geq 0. \end{aligned} \tag{1}$$

Determine the dual problem linked to (1). Write (1) and the dual problem in matrix form (with inequality constraints).

Solution. The dual problem can be written

$$\begin{aligned} &\min\ 50y_1 + 60y_2 + 5y_3 \\ &\text{subject to} \\ &y_1 + 4y_2 + y3 \geq 5, \\ &3y_1 + 2y_2 \geq 10, \\ &y_1, y_2, y_3 \geq 0. \end{aligned} \tag{2}$$

Let

$$c = \begin{pmatrix} 5 \\ 10 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 50 \\ 60 \\ 5 \end{pmatrix},$$

(Continued on page 2.)

The primal problem is then

$$\begin{aligned} &\max\ c^T x \\ &\text{subject to} \\ &Ax \leq b \\ &x \geq 0 \end{aligned} \tag{3}$$

and the dual is

$$\begin{aligned} &\min\ b^T y \\ &\text{subject to} \\ &y^T A \geq c^T \\ &y \geq 0 \end{aligned} \tag{4}$$

b

State the complementary slackness theorem for a general LP problem. Suppose $(x_1, x_2) = (5, 15)$ is an optimal solution to (1). Use the complementary slackness theorem to solve the dual problem of (1).

Solution.

Let x, w be a feasible solution for the primal and y, z for the dual problem respectively. These are optimal iff

$$x_j z_j = 0$$

for each j and

$$y_i w_i = 0$$

for each i. By inspection we see that $w_2 > 0$ and hence $y_2 = 0$. Moreover, since $x_1, x_2 > 0$ we have $z_1 = z_2 = 0$ and hence both dual constraints n (2) hold exactly. Using these we may solve for the remaining variables: $y_1 = 10/3$ and $y_3 = 5/3$.

c

(i) What is the definition of a convex set $C \subset \mathbb{R}^n$.

Solution. C is convex iff for all $x_1, x_2 \in C$ and $\lambda \in [0, 1]$

$$(1 - \lambda)x_1 + \lambda x_2 \in C.$$

(ii) Let $f : C \to \mathbb{R}$ be a convex continuous function. What does it mean (definition) that f is convex? Illustrate your definition with a figure.

Solution. f is convex iff for each $\lambda \in [0, 1]$

$$f((1 - \lambda)x_1 + \lambda x_2) \leq (1 - \lambda)f(x_1) + \lambda f(x_2)$$

See the figure below.

d

(i) Prove that a non-empty set $S \subset \mathbb{R}^n$ of optimal solutions to a LP problem is convex.

(Continued on page 3.)

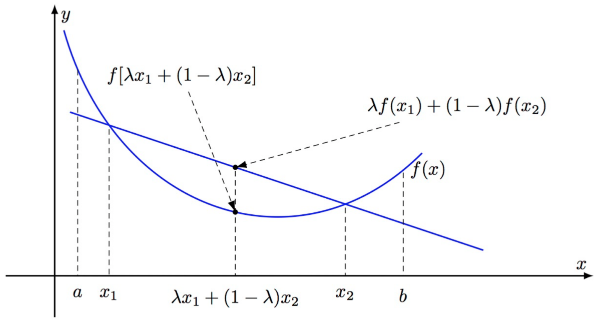

Figure 1: A convex function, in problem 1.c ii)

Solution. Let C denote the set of optimal solutions to an LP, i.e the set of $x \in \mathbb{R}^n$ that satisfies (1) with optimal value $c^T x = \eta$ for some $\eta \in \mathbb{R}$. Suppose $x_1, x_2 \in C$ and $\lambda \in [0,1]$ and consider

$$x = (1-\lambda)x_1 + \lambda x_2.$$

Since

$$Ax = (1-\lambda)Ax_1 + \lambda Ax_2 \leq (1-\lambda)b + \lambda b = b$$

and

$$c^T x = ((1-\lambda)c^T x_1 + \lambda c^T x_2) = (1-\lambda)\eta + \lambda\eta = \eta$$

we see that $x \in C$. Hence we may conclude that C is convex.
(ii) Consider a LP problem with two optimal solutions x^1 and x^2. Explain that this problem must in fact possess infinitely many optimal solutions.

Solution. Since C is convex, any $\lambda \in [0,1]$ yields a solution

$$x = (1-\lambda)x^1 + \lambda x^2 \in C.$$

Problem 2

a

Determine the pure minmax and maxmin strategies for the game given by

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 4 & -3 & 2 \\ 1 & -2 & -2 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

Does the game have a value? You must justify your answers (include definitions of the involved concepts).

Solution. The minmax strategy (for the row player) is to minimize the maximal payoff over all rows, i.e.

$$\min_i \max_j a_{ij} = 1$$

The maxmin strategy (for the column player) is to maximize the minimal payoff over all columns, i.e.

$$\max_j \min_i a_{ij} = 1.$$

Since the two match, we say that the game has a value, in this case 1

(Continued on page 4.)

b

Consider the game called *Odd-or-Even*. The row and column players simultaneously call out one of the numbers 1 or 2. The row player wins if the sum of the numbers is odd. The column player wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in kroner.

This is an example of a two-person zero-sum game, so that the payoff of the column player is the negative of the payoff of the row player. We will therefore restrict attention to the payoff matrix of the row player, which is denoted by A. For the Odd-or-Even game the payoff matrix becomes

$$A = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}, \tag{5}$$

where a negative amount means that the row player pays the absolute value of this amount to the column player.

What do we mean by a saddle point of a general game $A = \{a_{i,j}\} \in \mathbb{R}^{m \times n}$? Does the Odd-or-Even game (7) possess a saddle point?

Solution. A saddle point is an element a_{rs} of A for which

$$a_{rj} \leq a_{rs} \leq a_{is}$$

for all i, j, i.e. an element that is smallest in its column (s) and largest in its row (r). The Odd-Even game has no saddle-point,

c

We consider mixed (randomized) strategies $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of the Odd-or-Even game (7), given by two numbers $p, q \in (0, 1)$. The row player chooses $i = 1$ with probability $x_1 = p \in (0, 1)$. The column player chooses $j = 1$ with probability $y_1 = q \in (0, 1)$.

Solve the Odd-or-Even game by finding "equalizing strategies", that is, determine p such that if the row player chooses $i = 1$ with probability p, then the average payoff of the row player is the same whether the column player chooses $j = 1$ or $j = 2$. Compute the average payoff of the row player (with the probability p that you found).

Solution. We let $y = (p, 1 - p)$ and find

$$y^T A x = (p, 1-p) \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix} x = \begin{pmatrix} 3 - 5p & 7p - 4 \end{pmatrix} x \tag{6}$$

Since the expected payoff $y^T Ax$ should be indifferent to the column players choice x we require that $3 - 5p = 7p - 4$, hence $p = 7/12$. The average payoff is $3 - 5 * 7/12 = 1/12$.

Formulate the analogous principle for the column player, and use it to determine the probability q. Compute the average payoff of the column player (with the probability q that you found).

(Continued on page 5.)

Solution. We let $x = (q, 1-q)$ and find

$$y^T Ax = y^T \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix} (q, 1-q)^T = y^T \begin{pmatrix} 3-5q \\ 7q-4 \end{pmatrix} \tag{7}$$

Since the expected payoff $y^T Ax$ should be indifferent to the row players choice y we require that $3-5q = 7q-4$, hence $q = 7/12$. The average payoff is $3 - 5 * 7/12 = 1/12$.

Is the game is fair?

Solution. The game is not fair since the expected payoff (for the row player) is positive.

Problem 3

Consider the LP problem

$$\begin{aligned} &\max\ x_1 + 2x_3 \\ &\text{subject to} \\ &x_1 + 2x_2 + x_3 \leq 2, \\ &x_3 \leq 1, \\ &x_1, x_2, x_3 \geq 0. \end{aligned} \tag{8}$$

a

Use the simplex method to solve (8).

Solution. The first dictionary is

$$\begin{array}{l} \eta = x_1 + 2x_3 \\ \hline w_1 = 2 - x_1 - 2x_2 - x_3 \\ w_2 = 1 - x_3 \end{array} \tag{9}$$

which is primal feasible. Take x_3 into basis and w_2 out. This gives

$$\begin{array}{l} \eta = 2 + x_1 - 2w_2 \\ \hline w_1 = 1 - x_1 - 2x_2 + w_2 \\ x_3 = 1 - w_2 \end{array} \tag{10}$$

Take x_1 into basis and w_1 out. This gives

$$\begin{array}{l} \eta = 3 - w_1 - 2x_2 - w_2 \\ \hline x_1 = 1 - w_1 - 2x_2 + w_2 \\ x_3 = 1 - w_2 \end{array} \tag{11}$$

which is feasible and optimal. The solution is $x = (1, 0, 1)$ and the optimal value is 3.

b

Prove that if x^* and y^* are feasible for the primal and dual problems, respectively, and the corresponding objective values coincide, then x^* and y^* are optimal for their respective problems.

(Continued on page 6.)

Solution. From the constraints of the two problems we can deduce that

$$c^T x \leq y^T Ax \leq y^T b$$

for all feasible x and y. If $c^T x^* = b^T y^*$, this implies that there cannot be better feasible solutions for either problem.

c

Identify the dual problem linked to the LP problem

$$\begin{aligned} &\max 4x_1 + 5x_2 + 6x_3, \\ &x_1 + x_3 \leq 1, \\ &x_1 + x_2 \leq 2, \\ &x_2 + x_3 \leq 3, \\ &x_1, x_2, x_3 \geq 0. \end{aligned} \tag{12}$$

Without explicitly solving the problems, show that $(x_1, x_2, x_3) = (0, 2, 1)$ is the optimal solution of (12) and that $(y_1, y_2, y_3) = (\frac{5}{2}, \frac{3}{2}, \frac{7}{2})$ is the optimal solution of the dual problem.

Solution. The dual is

$$\begin{aligned} &\min y_1 + 2y_2 + 3y_3, \\ &y_1 + y_2 \geq 4, \\ &y_2 + y_3 \geq 5, \\ &y_1 + y_3 \geq 6, \\ &y_1, y_2, y_3 \geq 0. \end{aligned} \tag{13}$$

It is easily checked that (x_1, x_2, x_3) is primal feasible (satisfies the constraints) and (y_1, y_2, y_3) is dual feasible. The primal objective value is 16, which is identical to the dual objective value. These solutions are optimal by strong duality.

THE END