UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3100 — Linear Optimization
Day of examination:	Friday, June 7th, 2019
Examination hours:	09.00 – 13.00

This problem set consists of 7 pages.

Appendices:	None
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let

$$c = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 1 \\ 4 & 5 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 40 \\ 70 \\ 10 \end{pmatrix},$$

and consider the LP problem

$$\begin{aligned} &\max\ c^T x \\ &\text{subject to} \\ &Ax \le b \\ &x \ge 0 \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^2$.

a

Use the simplex method to find the solution (including slack variables) and optimal value of (1).

Solution. The first dictionary is

$$\begin{array}{l} \eta = 2x_1 + 3x_2 \\ \hline w_1 = 40 - 2x_1 - x_2 \\ w_2 = 70 - 4x_1 - 5x_2 \\ w_3 = 10 - x_2 \end{array} \tag{2}$$

This is feasible, so we can use the primal simplex method.. Take x_2 into basis and w_3 out. This gives

$$\begin{array}{l} \eta = 30 + 2x_1 - 3w_3 \\ \hline w_1 = 30 - 2x_1 + w_3 \\ w_2 = 20 - 4x_1 + 5w_3 \\ x_2 = 10 - w_3 \end{array} \tag{3}$$

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Take x_1 into basis and w_2 out. This gives

$$\begin{array}{l} \eta = 40 - 1/2w_2 - 1/2w_3 \\ \hline w_1 = 20 + 1/2w_2 - 3/2w_3 \\ x_1 = 5 - 1/4w_2 + 5/4w_3 \\ x_2 = 10 - w_3 \end{array} \tag{4}$$

which is feasible and optimal. The solution is $x = (5, 10)$ and $w = (20, 0, 0)$. The optimal objective value is 40.

b

Suppose (1) represents a resource allocation problem: a company uses three different raw materials to produce two different products; c_j is the profit per unit of product j ($j = 1, 2$); b_i is the number of units of raw material i in the company's warehouse ($i = 1, 2, 3$); a_{ij} is the number of units of raw material i required to produce 1 unit of product j ($i = 1, 2, 3$, $j = 1, 2$).

Introduce another company into the above situation, whose problem would naturally be to solve the dual of (1). Find the dual and explain the objective function and constraints in the dual problem from this company's perspective. Find the solution and optimal value of the dual problem.

Solution. Company B buys the raw materials from company A for the price of y_i per unit of material i. The objective function is to minimize the total cost and the constraints are that the price paid must make the deal at least as profitable for company A as producing and selling the products.

The dual is

$$\begin{array}{l} \min\ b^T y \\ \text{subject to} \\ A^T y \geq c \\ y \geq 0 \end{array} \tag{5}$$

where $y \in \mathbb{R}^3$. Using the negative transpose property on the optimal primal dictionary (found above), we obtain the dual dictionary

$$\begin{array}{l} -\xi = -40 - 20y_1 - 5z_1 - 10z_2 \\ \hline y_2 = 1/2 - 1/2y_1 + 1/4z_1 + 0z_2 \\ y_3 = 1/2 + 3/2y_1 - 5/4z_1 + 1z_2 \end{array} \tag{6}$$

From this we find a solution to the dual problem: $y = (0, 1/2, 1/2)$, $z = (0, 0)$ and the optimal value 40.

Alternative solution: by complementary slackness $y_1 = z_1 = z_2 = 0$. Hence both dual constraints hold exactly so, $2y_1 + 4y_2 = 2$ and $y_1 + 5y_2 + y_3 = 3$, which yields $y_2 = 1/2$ and $y_3 = 1/2$.

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Problem 2

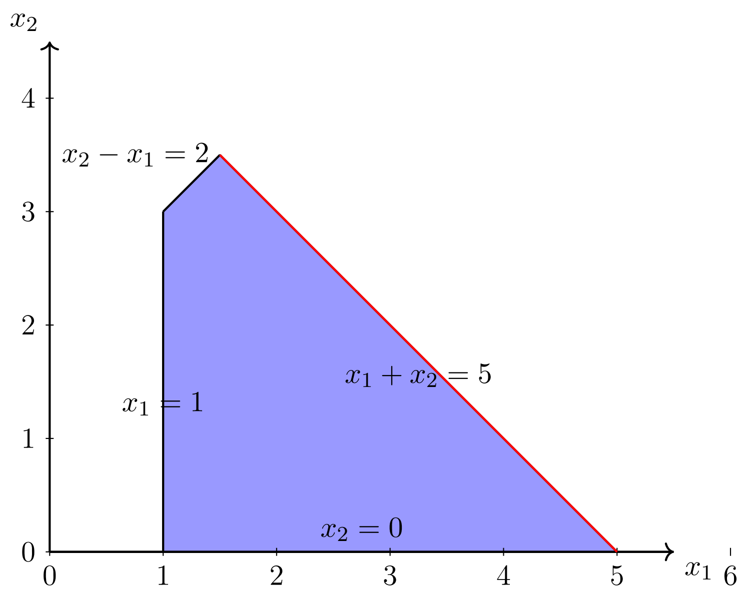
Consider the LP problem

$$\begin{aligned} &\max\ x_1 + x_2 \\ &\text{subject to} \\ &x_1 + x_2 \leq 5, \\ &x_2 - x_1 \leq 2, \\ &x_1 \geq 1, \\ &x_2 \geq 0. \end{aligned} \tag{7}$$

a

Draw the feasible region and answer the following: Is the basic solution with x_1, x_2 as non-basic variables feasible? Does the problem have a unique optimal solution? Find the optimal solution(s) graphically. Which variables are basic for the optimal solution(s)? Could cycling occur if one applied the simplex method to this problem? All answers must be justified.

Solution.



The basic solution with x_1, x_2 as non-basic variables, i.e. the point $(0, 0)$ is not feasible / not in the feasible region, as can be seen by inspection or by testing the inequalities. The line segment (marked in red) corresponding to the part of the feasible region which intersects with the line $x_1 + x_2 = 5$ is optimal, and hence there are infinitely many optimal solutions. The basic variables are those that corresponds to constraints that are not met exactly for an optimal solution, in this case x_1, x_2, w_3 or x_1, w_2, w_3. Cycling cannot occur because there are no degenerate solutions (with basic variables equal to zero), corresponding to more than two constraints beeing satisfied exactly.

b

Write (7) so that it involves a set of linear equations with slack variables w. Find the dictionary with x_1, x_2 and w_3 as basic variables. Show that the

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basic solution for this dictionary is feasible and optimal.

Solution. The equations are

$$\begin{aligned}
&\max\ x_1 + x_2\\
&\text{subject to}\\
&x_1 + x_2 + w_1 = 5,\\
&x_2 - x_1 + w_2 = 2,\\
&-x_1 + w_3 = -1,\\
&x_1, x_2, w_1, w_2, w_3 \geq 0.
\end{aligned} \tag{8}$$

We need to rearrange these equations so that they express the basic variables x_1, x_2, w_3 in terms of the nonbasic variables w_1, w_2. We can e.g. add the first and second equation to obtain

$$x_2 = 7/2 - 1/2w_1 - 1/2w_2$$

This can be substituted into the first equation to obtain

$$x_1 = 3/2 - 1/2w_1 + 1/2w_2$$

and finally, from the last equation

$$w_3 = 1/2 - 1/2w_1 + 1/2w_2$$

This results in the dictionary

$$\begin{aligned}
&\max 5 - w_1\\
&\text{subject to}\\
&x_1 = 3/2 - 1/2w_1 + 1/2w_2\\
&x_2 = 7/2 - 1/2w_1 - 1/2w_2\\
&w_3 = 1/2 - 1/2w_1 + 1/2w_2\\
&x_1, x_2, w_1, w_2, w_3 \geq 0.
\end{aligned} \tag{9}$$

which is clearly feasible and optimal, with optimal value 5.

c

State the complementary slackness theorem. Use this to solve the dual problem associated with (7).

Solution. Let x, w be a feasible solution for the primal and y, z for the dual problem respectively. These are optimal iff

$$x_j z_j = 0$$

for each j and

$$y_i w_i = 0$$

for each i. The dual can be written

$$\begin{aligned}
&\min\ 5y_1 + 2y_2 - y_3\\
&\text{subject to}\\
&y_1 - y_2 - y_3 \geq 1\\
&y_1 + y_2 \geq 1
\end{aligned} \tag{10}$$

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From the previous problem we have that $x_1, x_2, w_3 > 0$, which by the complementary slackness implies that $z_1 = z_2 = y_3 = 0$. This again implies that the two first dual constraints hold exactly for an optimal solution. Thus y_1, y_2 must satisfy

$$\begin{aligned} y_1 - y_2 &= 1 \\ y_1 + y_2 &= 1 \end{aligned} \tag{11}$$

which has the solution $y_1 = 1$ and $y_2 = 0$. Thus the dual solution is $y = (1, 0, 0)$ and $z = (0, 0)$, which is feasible and with objective value 5.

Problem 3

We consider the matrix game associated with the matrix

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 2 & 0 & 2 \\ -2 & -1 & 0 \end{pmatrix}.$$

a

Find the pure (deterministic) maxmin strategy for the column player and the corresponding pure minmax strategy for the row player and their corresponding objective values. Show that the game does not have a value for any pure strategies.

Solution. The pure maxmin strategy yields

$$\max_j \min_i a_{ij} = -1$$

The pure minmax strategy yields

$$\min_i \max_j a_{ij} = 0.$$

Since the two objective values are different the game does not have a value for any pure strategy.

b

State the Minimax Theorem for matrix games. Use this to explain why the pure strategies found above are not optimal, so that both players should consider using a mixed (probabilistic) strategy instead.

Solution. The minimax theorem can be stated as follows: There exist stochastic vectors x^* and y^* for which

$$\max_x y^{*T} A x = \min_y y^T A x^* \tag{12}$$

where x, y are stochastic vectors. This implies that there exists mixed strategies x^* and y^* that are mutually optimal. The pure strategies found above are special cases of mixed strategies. Since they dont have a common value, there is a "duality gap" which implies that they cannot both be optimal.

(Continued on page 6.)

c

Let $x^* = (0, \frac{1}{4}, \frac{3}{4})$ and $y^* = (\frac{1}{4}, 0, \frac{3}{4})$ be mixed (probabilistic) strategies for the column player and for the row player respectively. Prove that x^* and y^* are optimal strategies for the game associated with A. What is the value of the game? Which of the players would win in the long run?

Solution. We test the minimax theorem (12) on x^*, y^*. To that end

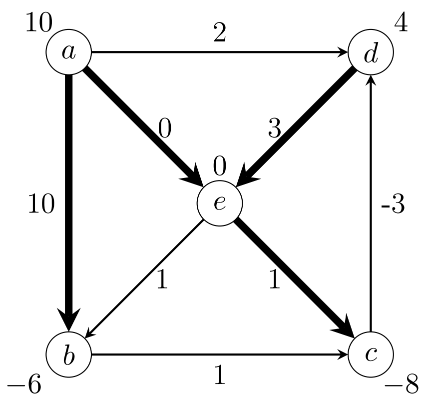
$$\min_y y^T A x^* = \min_y y^T(-1/4, 3/2, -1/4)^T = -1/4$$

and

$$\max_x y^* T A x = \max_x(-1, -1/4, -1/4)x = -1/4.$$

The two are equal, and therefore mutually optimal. The value of the game is $-1/4$, hence the row player wins in the long run.

Problem 4



Consider the minimum cost network flow problem based on the directed graph depicted in the figure. The number associated with a directed edge (i, j) is its cost c_{ij} (per unit flow), and the number associated with a node i is its supply b_i (negative values represent demands).

a

Explain why the subgraph indicated by the thick lines is a spanning tree. Compute the corresponding tree solution (all primal and dual variables).

Solution. The subgraph connects all the vertices and contains no cycles, hence it is a tree.

The tree solution is (the numbers associated with the vertices are the dual variables)

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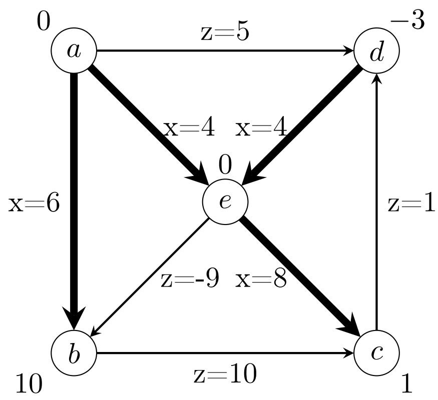


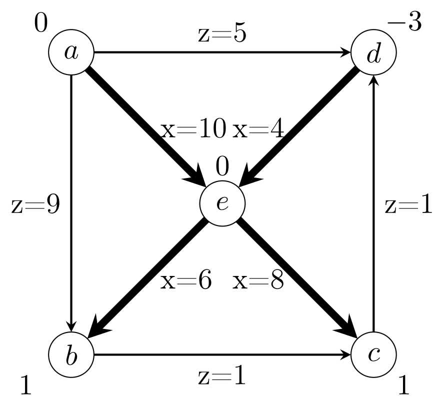
b

Use the primal network simplex algorithm to find an optimal solution (all primal and dual variables) and optimal value for the flow problem.

Solution. The solution in a) is infeasible as $z_{eb} = -9$ (negative value). Thus, we take x_{eb} into the basis. This yields a cycle on the vertices a, e, b. Adding a flow of 6 on this cycle in the direction of the edge eb, we see that the edge ab gets zero flow as the first one in the cycle and hence go out of basis. The primal flows are updated by adding a flow of 6 to the edges ae, eb, ba of the cycle. The only dual variables that needs updating is y_b, which is computed from node e in the new tree, with the value $y_b = 1$. The dual slack variables z_{ab} and z_{bc} are recomputed by e.g.

$$z_{ab} = c_{ab} - (y_b - y_a)$$

to obtain the new values $z_{ab} = 9$ and $z_{bc} = 1$, which are positive and hence feasible. Alternatively the updating can be done by adding 9 to the dual slack variables for edges not in the tree ending at vertex b and subtracting 9 from those starting at b.



The resulting tree solution is feasible with objective value

$$\eta = c^T x = 0 * 10 + 3 * 4 + 1 * 6 + 1 * 8 = 26.$$

THE END