UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT3100 — Linear optimization
Day of examination:	0900, 3 June 2020 – 0900, 10 June 2020

This problem set consists of 6 pages.

Appendices:	None
Permitted aids:	All

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 8 part questions will be weighted equally.

Problem 1 Simplex method

Consider the LP problem

$$\begin{array}{rrrl} \text{maximize} & -x_1 & +3x_2 & \\ \text{subject to} & -x_1 & +x_2 & \leq 1, \\ & x_1 & & \leq 4, \\ & & x_2 & \leq 3, \\ & x_1, x_2 \geq 0. & & \end{array}$$

1a

Solve this using the simplex method with initial feasible solution $(x_1, x_2) = (0, 0)$. Find an optimal solution and corresponding optimal objective value.

Answer: The initial dictionary is

$$\begin{array}{rcrrr} \eta & = & 0 & -x_1 & +3x_2 \\ \hline w_1 & = & 1 & +x_1 & -x_2 \\ w_2 & = & 4 & -x_1 & \\ w_3 & = & 3 & & -x_2 \end{array}$$

x_2 enters the basis, w_1 leaves:

$$\begin{array}{rcrrr} \eta & = & 3 & +2x_1 & -3w_1 \\ \hline x_2 & = & 1 & +x_1 & -w_1 \\ w_2 & = & 4 & -x_1 & \\ w_3 & = & 2 & -x_1 & +w_1 \end{array}$$

x_1 enters the basis, w_3 leaves:

(Continued on page 2.)

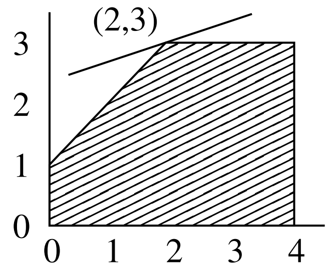
$$\begin{array}{rcllll} \eta & = & 7 & -2w_3 & -w_1 \\ \hline x_2 & = & 3 & -w_3 & \\ w_2 & = & 2 & +w_3 & -w_1 \\ x_1 & = & 2 & -w_3 & +w_1 \end{array}$$

This dictionary is optimal. The optimal solution is $(x_1, x_2) = (2, 3)$ (and $w_1 = w_3 = 0$, $w_2 = 2$) with $\eta = 7$.

1b

Illustrate the problem geometrically. Draw the feasible region and the contour line of the objective function $f(x_1, x_2) = -x_1 + 3x_2$ that passes through the optimal solution.

Answer: The feasible region is the 5-sided polygon with vertices $(0, 0)$, $(4, 0)$, $(4, 3)$, $(2, 3)$, $(0, 1)$. The contour line is the straight line passing through the vertex $(2, 3)$ perpendicular to the vector $(-1, 3)$.



Problem 2 Standard form

Convert the LP problem

$$\begin{array}{rllll} \text{maximize} & 3x_1 & +2x_2 & +4x_3 & \\ \text{subject to} & 2x_1 & -5x_2 & & = 6, \\ & -x_1 & & +3x_3 & \geq 4, \\ & & x_1, x_2 \geq 0 & & \end{array}$$

into standard form (the form suitable for the simplex algorithm). Note that $x_3 \in \mathbb{R}$ is a free variable. What form of the simplex algorithm will be required to solve it? (do not try to solve it).

Answer: We can convert the constraints as follows:

$$\begin{array}{rllll} \text{maximize} & 3x_1 & +2x_2 & +4x_3 & \\ \text{subject to} & 2x_1 & -5x_2 & & \leq 6, \\ & -2x_1 & +5x_2 & & \leq -6, \\ & x_1 & & -3x_3 & \leq -4, \\ & & x_1, x_2 \geq 0. & & \end{array}$$

To deal wih the free variable x_3 we let $x_3 = y_3 - y_4$ for $y_3, y_4 \geq 0$. Then the standard form is

(Continued on page 3.)

$$\begin{array}{lrrrrl} \text{maximize} & 3x_1 & +2x_2 & +4y_3 & -4y_4 & \\ \text{subject to} & 2x_1 & -5x_2 & & & \leq 6, \\ & -2x_1 & +5x_2 & & & \leq -6, \\ & x_1 & & -3y_3 & +3y_4 & \leq -4, \\ & & x_1, x_2, y_3, y_4 \geq 0. & & & \end{array}$$

The 2-phase simplex method will be required because the right hand side is not non-negative.

Problem 3 Duality

Consider the LP problem

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j, \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \ldots, m, \\ & x_j \geq 0, \quad j = 1, 2, \ldots, n, \end{aligned}$$

and its dual

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{m} b_i y_i, \\ \text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, \ldots, n, \\ & y_i \geq 0, \quad i = 1, 2, \ldots, m. \end{aligned}$$

3a

Let w_i be the i-th slack variable in the primal problem, $i = 1, 2, \ldots, m$, and let z_j be the j-th slack variable in the dual problem, $j = 1, 2, \ldots, n$. Derive the following identity:

$$\sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} w_i y_i + \sum_{j=1}^{n} z_j x_j, \tag{1}$$

and use it to prove the Weak Duality Theorem.

Answer: The slack variables are

$$\begin{aligned} w_i &:= b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m, \\ z_j &:= \sum_{i=1}^{m} a_{ij} y_i - c_j, \quad j = 1, 2, \ldots, n. \end{aligned}$$

(Continued on page 4.)

Then

$$\sum_{i=1}^{m} b_i y_i - \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j + w_i \right) y_i - \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i - z_j \right) x_j$$
$$= \sum_{i=1}^{m} w_i y_i + \sum_{j=1}^{n} z_j x_j.$$

The weak duality theorem states that if $x = (x_1, x_2, \ldots, x_n)$ is feasible for (P) and $y = (y_1, y_2, \ldots, y_m)$ is feasible for (D) then

$$\sum_j c_j x_j \leq \sum_i b_i y_i.$$

To prove it, suppose x is feasible for (P) and y is feasible for (D). Then all the variables x_j, y_i, z_j, w_i are non-negative and the right hand side of (1) is non-negative. Therefore,

$$\sum_i b_i y_i - \sum_j c_j x_j \geq 0.$$

3b

Recall that the Strong Duality Theorem states that if (P) has an optimal solution x^* then (D) has an optimal solution y^* and that

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*.$$

State the Complementary Slackness Theorem and and prove it *using the identity* (1).

Answer: The complementary slackness theorem says that feasible x and y are optimal if and only if

$$\begin{aligned} x_j z_j &= 0, \quad j = 1, 2, \ldots, n, \\ y_i w_i &= 0, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{2}$$

To prove it, suppose x and y are optimal. Then by (1),

$$\sum_i w_i y_i + \sum_j z_j x_j = 0.$$

Then all the products $x_j z_j$ and $y_i w_i$, being non-negative, must be zero.

Conversely, suppose that (2) holds. Then by (1),

$$\sum_i b_i y_i - \sum_j c_j x_j = 0.$$

(Continued on page 5.)

3c

What is the optimal solution to the dual of the LP problem in Problem 1?

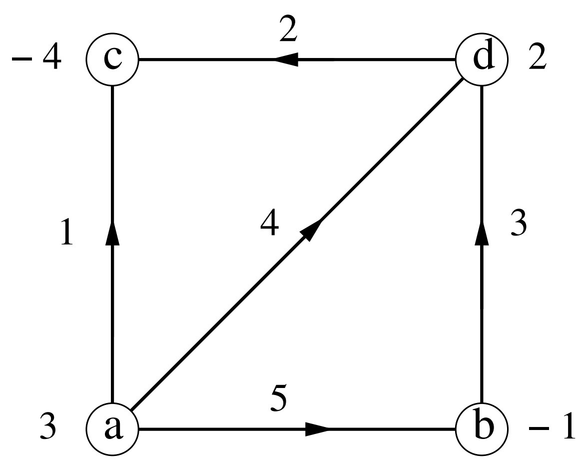
Answer: The optimal dual dictionary is the negative transpose of the optimal primal dictionary, which, from Problem 1a, is

$$\begin{array}{rcrrrr} -\eta & = & -7 & -3z_2 & -2y_2 & -2z_1 \\ \hline y_3 & = & 2 & +z_2 & -y_2 & +z_1 \\ y_1 & = & 1 & & +y_2 & -z_1 \end{array}$$

So the optimal solution is $(y_1, y_2, y_3) = (1, 0, 2)$ (and $z_1 = z_2 = 0$) with $\eta = 7$ as in the primal solution.

Problem 4 Network flow

Consider the minimum cost network flow problem based on the directed graph shown in the figure. The number associated with each directed edge



(i, j) is its cost $c_{i,j}$ (per unit flow). The number associated with each node i is the supply b_i.

4a

Let T_1 be the spanning tree consisting of the edges

$$(a, b), \quad (a, d), \quad (d, c).$$

Compute the tree solution x corresponding to T_1.

Answer: The flow balance equation at node i is

$$\text{sum of outflow} - \text{sum of inflow} = b_i.$$

By 'tree solution' we mean that there is zero flow on edges not in T_1, i.e.,

$$x_{ac} = x_{bd} = 0.$$

Using leaf elimination, for example, in the given order of the three edges of T_1, we use the supplies b_i to obtain

$$x_{ab} = 1, \quad x_{ad} = 2, \quad x_{dc} = 4.$$

(Continued on page 6.)

4b

Use the network simplex method to find an optimal solution and optimal value for the flow problem.

Answer: x above is a feasible solution. We compute the dual variables using $y_j = y_i + c_{ij}$ for each edge (i,j) in T_1. Use node a as the root and set $y_a = 0$. Then

$$y_a = 0, \quad y_b = 5, \quad y_c = 6, \quad y_d = 4.$$

We now compute the dual slacks $z_{ij} = c_{ij} - (y_j - y_i)$ on the edges (i,j) not in T_1:

$$z_{ac} = -5, \quad z_{bd} = 4.$$

Since z_{ac} is negative, x is not an optimal solution. So, we pivot. We take x_{ac} into the basis. If we increase x_{ac} from 0 to ϵ, then from the supplies, the new flows are as before, except in the cycle $(a,c),(c,d),(b,d)$ where they are

$$x_{ac} = \epsilon, \quad x_{cd} = 4 - \epsilon, \quad x_{ad} = 2 - \epsilon.$$

The maximum allowed increase in x_{ac} is therefore $\epsilon = 2$, and this makes $x_{ad} = 0$, and so x_{ad} leaves the basis. This gives us a new spanning tree T_2 with edges

$$(a,b), \quad (a,c), \quad (d,c),$$

and the new tree solution x is given by

$$x_{ad} = x_{bd} = 0,$$

and

$$x_{ab} = 1, \quad x_{ac} = 2, \quad x_{cd} = 2.$$

The dual variables, with $y_a = 0$, are

$$y_a = 0, \qquad y_b = 5, \quad y_c = 1, \quad y_d = -1,$$

and then

$$z_{ad} = 5, \quad z_{bd} = 9.$$

Now all the z_{ij} are non-negative and so x is an optimal solution. The optimal objective value (minimum cost) is

$$\sum_{(i,j)\in T_2} c_{ij}x_{ij} = c_{ab}x_{ab} + c_{ac}x_{ac} + c_{cd}x_{cd} = 5 \times 1 + 1 \times 2 + 2 \times 2 = 11.$$

Good luck!