UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3100 — Linear optimization.

Day of examination: Wednesday 16. June 2021.

Examination hours: 09:00 – 13:00.

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: All.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 (Simplex method). We will consider the following LP.

$$\begin{array}{lrcrcr} \max & -x_1 & - & 2x_2 & & \\ \text{s.t.} & -x_1 & - & x_2 & \leq & -2 \\ & x_1 & - & x_2 & \leq & 2 \\ & -x_1 & + & x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 6 \\ & & & x_1, x_2 & \geq & 0 \end{array} \tag{1}$$

a) Write down the dual problem. Write also both the primal and dual problems in matrix form.

Solution: The dual problem is

$$\begin{array}{lrcrcrcrcr} \min & -2y_1 & + & 2y_2 & + & 2y_3 & + & 6y_4 & & \\ \text{subject to} & -y_1 & + & y_2 & - & y_3 & + & y_4 & \geq & -1 \\ & -y_1 & - & y_2 & + & y_3 & + & y_4 & \geq & -2 \\ & & & & & & & y_1, y_2, y_3, y_4 & \geq & 0 \end{array}$$

The primal and dual problems can be written as

$$\begin{array}{lrcl} \max & \mathbf{c}^T\mathbf{x} & & \\ \text{s.t.} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

and

$$\begin{array}{lrcl} \min & \mathbf{b}^T\mathbf{y} & & \\ \text{s.t.} & A^T\mathbf{y} & \geq & \mathbf{c} \\ & \mathbf{y} & \geq & \mathbf{0} \end{array}$$

where

$$A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -2 \\ 2 \\ 2 \\ 6 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

(Continued on page 2.)

b) Draw the feasible region of (1).
Solution: The feasible region is the square with vertices $(2,0)$, $(0,2)$, $(2,4)$, and $(4,2)$. Note that this square does not contain the origin, which implies that the initial primal dictionary is not feasible.

c) Write down the (initial) primal dictionary and the corresponding dual dictionary. Explain that the primal dictionary is not feasible, while the dual dictionary is.
Solution: The primal dictionary is

$$
\begin{array}{rcrcrcr}
\eta & = & & - & x_1 & - & 2x_2 \\
\hline
w_1 & = & -2 & + & x_1 & + & x_2 \\
w_2 & = & 2 & - & x_1 & + & x_2 \\
w_3 & = & 2 & + & x_1 & - & x_2 \\
w_4 & = & 6 & - & x_1 & - & x_2.
\end{array}
$$

The dual dictionary is obtained by taking the negative transpose:

$$
\begin{array}{rcrcrcrcrcr}
-\zeta & = & & & 2y_1 & - & 2y_2 & - & 2y_3 & - & 6y_4 \\
\hline
z_1 & = & 1 & - & y_1 & + & y_2 & - & y_3 & + & y_4 \\
z_2 & = & 2 & - & y_1 & - & y_2 & + & y_3 & + & y_4
\end{array}
$$

We see that the initial primal dictionary is not feasible (since the constant in the first constraint is negative), while the initial dual dictionary is (since both constants in the constraints are positive).

d) Apply the simplex method to the dual problem, and write down the corresponding optimal dictionary for the primal problem. What are the optimal solutions to the primal and dual problems? Are they unique?
Solution: In the dual dictionary y_1 is the only possible choice for entering variable. The ratios are 1 and $1/2$, so that z_1 is leaving. We rewrite $z_1 = 1 - y_1 + y_2 - y_3 + y_4$ as $y_1 = 1 - z_1 + y_2 - y_3 + y_4$, and insert this to obtain

$$
\begin{array}{rcrcrcrcrcr}
-\zeta & = & 2 & - & 2z_1 & & & - & 4y_3 & - & 4y_4 \\
\hline
y_1 & = & 1 & - & z_1 & + & y_2 & - & y_3 & + & y_4 \\
z_2 & = & 1 & + & z_1 & - & 2y_2 & + & 2y_3 & &
\end{array}
$$

This dictionary is optimal. The corresponding basic solution is $\mathbf{y} = (1,0,0,0)$. It is not unique since we can increase y_2 from 0 to $1/2$ without affecting feasibility, and without affecting the objective value. The general solution to the dual problem is thus

$$(1 + y_2, y_2, 0, 0), \text{ for } 0 \leq y_2 \leq 1/2.$$

The corresponding optimal primal dictionary is

$$
\begin{array}{rcrcrcr}
\eta & = & -2 & - & w_1 & - & x_2 \\
\hline
x_1 & = & 2 & + & w_1 & - & x_2 \\
w_2 & = & & - & w_1 & + & 2x_2 \\
w_3 & = & 4 & + & w_1 & - & 2x_2 \\
w_4 & = & 4 & - & w_1 & &
\end{array}
$$

The corresponding basic optimal solution to the primal problem is $\mathbf{x} = (2,0)$. It is unique.

(Continued on page 3.)

e) What is the optimal (x_1, x_2) if the objective in (1) is changed to $-x_1 - x_2$? Is the optimal solution unique?
Solution: We rewrite the objective as

$$-x_1 - x_2 = -(2 + w_1 - x_2) - x_2 = -2 - w_1.$$

The corresponding primal dictionary is thus

$$\begin{array}{rcrcrcr}
\eta & = & -2 & - & w_1 & & \\
\hline
x_1 & = & 2 & + & w_1 & - & x_2 \\
w_2 & = & & - & w_1 & + & 2x_2 \\
w_3 & = & 4 & + & w_1 & - & 2x_2 \\
w_4 & = & 4 & - & w_1 & &
\end{array}$$

which also is optimal. The corresponding basic solution is $\mathbf{x} = (2, 0)$. This is not unique since we can increase x_2 without affecting the objective value. We can't increase x_2 to more than 2, however, in order to maintain feasibility. The optimal solutions are thus the line segment $(2 - x_2, x_2)$ for $0 \leq x_2 \leq 2$, i.e., the line segment from $(2, 0)$ to $(0, 2)$.

Problem 2 (Convexity).
What does it mean that a set $C \subseteq \mathbb{R}^n$ is convex, and that a function f from C to $\mathbb{R}$ is convex?
Show also that, if f is a convex function, then $h(x) = e^{f(x)}$ is also convex.
Hint: Use that $g(x) = e^x$ also is convex (you can use this fact without proving it).
Solution: A set C is convex if $(1 - \lambda)x + \lambda y \in C$ whenever $x \in C$ and $y \in C$. A function f is convex if $f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y)$ for all $x, y \in C$.
Since f is convex we have that

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y).$$

Since e^x is increasing we have that

$$e^{f((1-\lambda)x+\lambda y)} \leq e^{(1-\lambda)f(x)+\lambda f(y)}.$$

We now obtain

$$\begin{aligned}
h((1 - \lambda)x + \lambda y) &= e^{f((1-\lambda)x+\lambda y)} \\
&\leq e^{(1-\lambda)f(x)+\lambda f(y)} = g((1 - \lambda)f(x) + \lambda f(y)).
\end{aligned}$$

The hint says that $g(x)$ is convex, so that

$$\begin{aligned}
g((1 - \lambda)f(x) + \lambda f(y)) &\leq (1 - \lambda)g(f(x)) + \lambda g(f(y)) \\
&= (1 - \lambda)h(x) + \lambda h(y).
\end{aligned}$$

It follows that h is convex.

Problem 3 (Game theory). Consider the matrix game with payoff matrix $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$. In the following x denotes a (randomized) strategy for the column player, y a (randomized) strategy for the row player.

(Continued on page 4.)

a) This question has two parts:

(i): Assume that the row player chooses strategy $y^* = (1, 0)$ (i.e., he always chooses the first item). What is the optimal strategy x for the column player (in order to maximize payoffs to himself)? What is the corresponding expected payoff?

Solution: We need to solve

$$\max_{x \geq 0, \mathbf{1}^T x = 1} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} x = \max_{x \geq 0, \mathbf{1}^T x = 1} x_1 - x_2.$$

The maximum is clearly attained for $x = (1, 0)$, with a maximum/expected payoff of 1.

(ii): Assume that the column player chooses strategy $x^* = (1, 0)$. What is the optimal strategy y for the row player (in order to minimize payoffs from himself)? What is the corresponding expected payoff?

Solution: We need to solve

$$\min_{y \geq 0, \mathbf{1}^T y = 1} y^T \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \min_{y \geq 0, \mathbf{1}^T y = 1} y_1 + 2y_2.$$

The minimum is clearly attained for $y = (1, 0)$, with a minimum/expected payoff of 1.

b) Is it possible for the row player to choose a better strategy than y^*, i.e., so that his expected payoff is lower than what you obtained above?

Solution: The strategy y^* is optimal.

There are several ways to argue for this. One way is to realize this in terms of pure minmax/maxmin strategies (not in the syllabus).

One can also argue as follows. If the second item is chosen with some probabiblity $y_2 > 0$, and x always chooses the first item (i.e., $x = (1, 0)$), the payoff would be

$$\begin{pmatrix} y_1 & y_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = y_1 + 2y_2 = 1 + y_2 > 1,$$

so that y gives higher payoff when compared to y^* (i.e., is suboptimal).

One may argue in similar ways with the minimax theorem. If a better strategy y exists, we would have

$$\max_{x \geq 0, \mathbf{1}^T x = 1} y^T A x < \max_{x \geq 0, \mathbf{1}^T x = 1} (y^*)^T A x. \tag{2}$$

But we have that

$$\begin{aligned} 1 &= \min_{y \geq 0, \mathbf{1}^T y = 1} y^T A x^* \\ &\leq \max_{x \geq 0, \mathbf{1}^T x = 1} \min_{y \geq 0, \mathbf{1}^T y = 1} y^T A x \\ &= \min_{y \geq 0, \mathbf{1}^T y = 1} \max_{x \geq 0, \mathbf{1}^T x = 1} y^T A x \\ &\leq \max_{x \geq 0, \mathbf{1}^T x = 1} (y^*)^T A x = 1 \end{aligned}$$

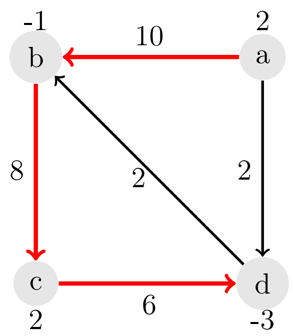
where we used a), and where the middle equality was proved in chapter 11. It follows that all of them are equal, so that

$$\min_{y \geq 0, \mathbf{1}^T y = 1} \max_{x \geq 0, \mathbf{1}^T x = 1} y^T A x = \max_{x \geq 0, \mathbf{1}^T x = 1} (y^*)^T A x,$$

but this contradicts (2).

(Continued on page 5.)

Problem 4 (Network flow). Consider the minimum cost network flow problem based on the directed graph shown in the figure below.



The number associated with each directed edge is the cost per unit flow, and the number associated with each node is the supply at that node.

a) Let T_1 be the spanning tree consisting of the edges (a, b), (b, c), and (c, d) (this is indicated in red above). Compute the tree solution corresponding to T_1.

Solution: We apply the flow balance equations at the nodes a, b, and c.

- Flow balance at a gives $x_{ab} = 2$.
- Flow balance at b gives $x_{bc} - x_{ab} = -1$, so that $x_{bc} = 1$.
- Flow balance at c gives $x_{cd} - x_{bc} = 2$, so that $x_{cd} = 3$.

We also have that $x_{db} = x_{ad} = 0$, since (d, b) and (a, d) are outside T_1. Note that the tree solution we found is (primal) feasible.

b) Use the network simplex method to find an optimal solution and optimal value for the flow problem.

Solution: Let a be the root node, so that $y_a = 0$. We first find the dual variables. We apply that $y_j - y_i = c_{ij}$ for each $(i, j) \in T_1$:

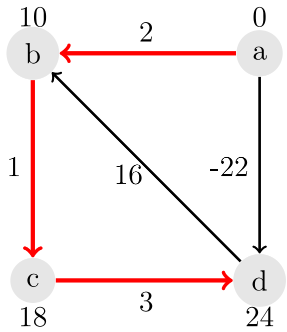
- $(i, j) = (a, b)$: $y_b - y_a = 10$, so that $y_b = 10$.
- $(i, j) = (b, c)$: $y_c - y_b = 8$, so that $y_c = 18$.
- $(i, j) = (c, d)$: $y_d - y_c = 6$, so that $y_d = 24$.

For the dual slack variables we apply that $z_{ij} = y_i + c_{ij} - y_j$ for each $(i, j) \notin T_1$:

- $(i, j) = (d, b)$: $z_{db} = y_d + c_{db} - y_b = 24 + 2 - 10 = 16$.
- $(i, j) = (a, d)$: $z_{ad} = y_a + c_{ad} - y_d = 0 + 2 - 24 = -22$.

This gives the following graph, where the supplies have beed replaced by dual variables, and costs have been replaced by flows (for arcs in the spanning tree) or dual slacks (for arcs outside the spanning tree):

(Continued on page 6.)



Since $z_{ad} < 0$ we do not have dual feasibility, and we thus let x_{ad} enter the basis, and increase it by ϵ. This gives a loop involving the four edges (a, b), (b, c), (c, d), and (a, d). Due to the directions of the edges and flow balance, the flow is changed as follows:

$$\begin{aligned}\tilde{x}_{ab} &= x_{ab} - \epsilon = 2 - \epsilon\\ \tilde{x}_{bc} &= x_{bc} - \epsilon = 1 - \epsilon\\ \tilde{x}_{cd} &= x_{cd} - \epsilon = 3 - \epsilon\\ \tilde{x}_{ad} &= \epsilon\end{aligned}$$

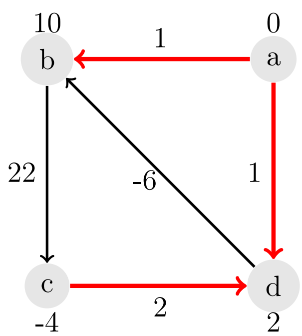
We see that $\tilde{x}_{bc}$ becomes zero first, and that this occurs for $\epsilon = 1$. x_{bc} thus leaves the basis. The new spanning tree is thus $T_2 = \{(a, b), (c, d), (a, d)\}$, and the new tree solution is $x_{ab} = 1$, $x_{cd} = 2$, $x_{ad} = 1$.
We now update the dual variables by applying $y_j - y_i = c_{ij}$ for each $(i, j) \in T_2$:

- $(i, j) = (a, b)$: $y_b - y_a = 10$, so that $y_b = 10$.
- $(i, j) = (a, d)$: $y_d - y_a = 2$, so that $y_d = 2$.
- $(i, j) = (c, d)$: $y_d - y_c = 6$, so that $y_c = -4$.

For the dual slack variables we apply that $z_{ij} = y_i + c_{ij} - y_j$ for each $(i, j) \notin T_2$:

- $(i, j) = (b, c)$: $z_{bc} = y_b + c_{bc} - y_c = 10 + 8 + 4 = 22$.
- $(i, j) = (d, b)$: $z_{db} = y_d + c_{db} - y_b = 2 + 2 - 10 = -6$.

This gives the following graph, where T_2 is coloured in red:



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Since $z_{db} < 0$ we do not have dual feasibility, and we thus let x_{db} enter the basis, and increase it by ϵ. This gives a loop involving the edges (a, b), (a, d), and (d, b). Due to the directions of the edges and flow balance, the flow is changed as follows:

$$\begin{aligned}\tilde{x}_{ab} &= x_{ab} - \epsilon = 1 - \epsilon \\ \tilde{x}_{ad} &= x_{ad} + \epsilon = 1 + \epsilon \\ \tilde{x}_{db} &= \epsilon\end{aligned}$$

We see that $\tilde{x}_{ab}$ becomes zero first, and that this occurs for $\epsilon = 1$. x_{ab} thus leaves the basis. The new spanning tree is thus $T_3 = \{(a, d), (d, b), (c, d)\}$ ((c, d) stays in the spanning tree, it was outside the loop. Its flow does not change), and the new tree solution is $x_{ad} = 2$, $x_{db} = 1$, $x_{cd} = 2$.

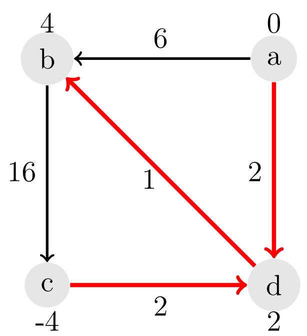
We now update the dual variables by applying $y_j - y_i = c_{ij}$ for each $(i, j) \in T_3$:

- $(i, j) = (a, d)$: $y_d - y_a = 2$, so that $y_d = 2$.
- $(i, j) = (d, b)$: $y_b - y_d = 2$, so that $y_b = 4$.
- $(i, j) = (c, d)$: $y_d - y_c = 6$, so that $y_c = -4$.

For the dual slack variables we apply that $z_{ij} = y_i + c_{ij} - y_j$ for each $(i, j) \notin T_3$:

- $(i, j) = (a, b)$: $z_{ab} = y_a + c_{ab} - y_b = 0 + 10 - 4 = 6$.
- $(i, j) = (b, c)$: $z_{bc} = y_b + c_{bc} - y_c = 4 + 8 + 4 = 16$.

This gives the following graph, where T_3 is coloured in red:



We see that we now have dual feasibility. The current x is thus optimal, and the objective value is

$$c_{ad}x_{ad} + c_{db}x_{db} + c_{cd}x_{cd} = 2 \cdot 2 + 2 \cdot 1 + 6 \cdot 2 = 4 + 2 + 12 = 18.$$

Good luck!