UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in:	MAT3100 — Linear optimization.
Day of examination:	Monday 13. June 2022.
Examination hours:	15:00 – 19:00.

This problem set consists of 2 pages.

Appendices:	None.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 (Simplex method). We will consider the following linear programming problem.

$$\begin{array}{lrcrcl} \max & x_1 & + & 2x_2 & & \\ \text{s.t.} & -2x_1 & + & x_2 & \leq & 0 \\ & 2x_1 & - & x_2 & \leq & 4 \\ & & & x_2 & \leq & 2 \\ & & & x_1, x_2 & \geq & 0 \end{array} \tag{1}$$

a) Draw the feasible region of (1).

b) Write down the dictionary corresponding to (1), and solve the problem using the simplex method. Write also down the optimal value and optimal solution.

c) Write down the dual problem and the optimal dual dictionary. What is the optimal solution to the dual problem?

d) What is the optimal solution if the objective in (1) is changed to $6x_1 - 3x_2$? Is the optimal solution unique?

Problem 2 (Convexity).

a) Let $f : [a, b] \to \mathbb{R}$ be a convex function. Show that

$$\max\{f(x) : x \in [a, b]\} = \max\{f(a), f(b)\}$$

In other words, show that a convex function defined on an interval on the real line achieves its maximum in one of the end points of that interval.

b) Let $C \subseteq \mathbb{R}^n$ be a convex set and consider the function d_C defined by $d_C(x) = \inf\{\|x - c\| : c \in C\}$ (i.e., the smallest distance from x to C). Show that d_C is a convex function.
Hint: For points x, y, the point $(1 - \lambda)x_1 + \lambda y_1$ can be useful here, where x_1 and y_1 are points in C near to achieving the (minimal) distances from x and y to C.

(Continued on page 2.)

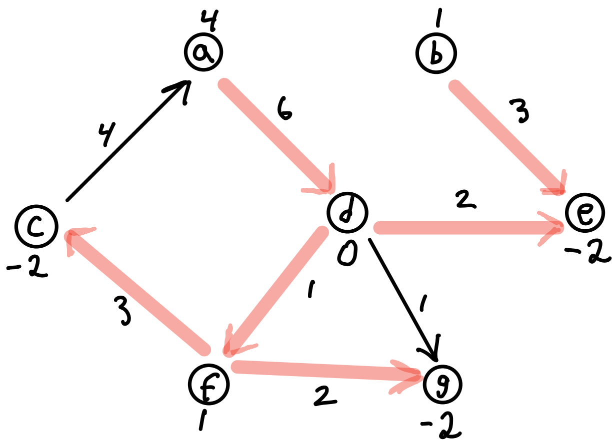

Figure 1: Flow problem for Problem 4.

Problem 3 (Game theory). Consider the matrix game with payoff matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 \\ -1 & -2 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

In the following x denotes a (randomized) strategy for the column player, y a (randomized) strategy for the row player.

a) This question has two parts:

(i): Assume that the row player chooses strategy $y^* = (1/2, 1/2, 0, 0)$ (i.e., he chooses the first and second items with equal probability, and never the two others). What is the optimal strategy x for the column player (in order to maximize payoffs to himself)? What is the expected payoff?

(ii): Assume that the column player chooses strategy $x^* = (0, 0, 1/2, 1/2)$ (i.e., he chooses the third and fourth items with equal probability, and never the two others). What is the optimal strategy y for the row player (in order to minimize payoffs from himself)? What is the expected payoff?

b) Let us instead consider the strategy $y^* = (1/3, 1/3, 0, 1/3)$ for the row player, and the strategy $x^* = (1/3, 0, 1/3, 1/3)$ for the column player. Are these strategies mutually optimal? If so, what is the value of the game?

Problem 4 (Network flow). Consider the minimum cost network flow problem based on the directed graph shown in Figure 1.

The number associated with each directed edge is the cost per unit flow, and the number associated with each node is the supply at that node.

a) Let T_1 be the spanning tree consisting of the edges (a, d), (d, e), (b, e), (d, f), (f, g), and (f, c) (indicated in bold in the figure). Compute the tree solution corresponding to T_1.

b) Use the network simplex method to find an optimal solution and optimal value for the flow problem.

Good luck!