UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3100 — Linear optimization.

Day of examination: Monday 13. June 2022.

Examination hours: 15:00 – 19:00.

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 (Simplex method). We will consider the following linear programming problem.

$$
\begin{array}{lrcrcl}
\max & x_1 & + & 2x_2 & & \\
\text{s.t.} & -2x_1 & + & x_2 & \leq & 0 \\
 & 2x_1 & - & x_2 & \leq & 4 \\
 & & & x_2 & \leq & 2 \\
 & & & x_1, x_2 & \geq & 0
\end{array} \tag{1}
$$

a) Draw the feasible region of (1).
Solution: The feasible region has corners $(0,0)$, $(2,0)$, $(3,2)$, $(1,2)$.

b) Write down the dictionary corresponding to (1), and solve the problem using the simplex method. Write also down the optimal value and optimal solution.
Solution: The dictionary is

$$
\begin{array}{rclcrcr}
\eta & = & & & x_1 & + & 2x_2 \\
w_1 & = & & & 2x_1 & - & x_2 \\
w_2 & = & 4 & - & 2x_1 & + & x_2 \\
w_3 & = & 2 & & & - & x_2
\end{array}
$$

If we choose x_1 as the entering variable, the ratios become $-\infty$, $1/2$, and 0, so that w_2 leaves. We substitute $x_1 = 2 - \frac{1}{2}w_2 + \frac{1}{2}x_2$, and get the new dictionary

$$
\begin{array}{rclcrcr}
\eta & = & 2 & - & \frac{1}{2}w_2 & + & \frac{5}{2}x_2 \\
w_1 & = & 4 & - & w_2 & & \\
x_1 & = & 2 & - & \frac{1}{2}w_2 & + & \frac{1}{2}x_2 \\
w_3 & = & 2 & & & - & x_2
\end{array}
$$

Now x_2 enters, and the ratios are 0, $-1/4$, and $1/2$, so that w_3 leaves. We

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substitute $x_2 = 2 - w_3$, and get the new dictionary

$$\begin{array}{rcrcrcr} \eta & = & 7 & - & \frac{1}{2}w_2 & - & \frac{5}{2}w_3 \\ w_1 & = & 4 & - & w_2 & & \\ x_1 & = & 3 & - & \frac{1}{2}w_2 & - & \frac{1}{2}w_3 \\ x_2 & = & 2 & & & - & w_3 \end{array}$$

This dictionary is optimal. The optimal value is 7, and $x = (3, 2)$ is the unique optimal solution.

It is also possible here to let x_2 enter in the first pivot (this is the choice taken by the largest coefficient rule). In this case three pivots will be needed in total, and the first pivot is degenerate.

c) Write down the dual problem and the optimal dual dictionary. What is the optimal solution to the dual problem?
Solution: The dual problem is

$$\begin{array}{lrcrcrcr} \min & & & 4y_2 & + & 2y_3 & & \\ \text{subject to} & -2y_1 & + & 2y_2 & & & \geq & 1 \\ & y_1 & - & y_2 & + & y_3 & \geq & 2 \\ & & & & & y_1, y_2, y_3 & \geq & 0 \end{array}$$

The optimal dual dictionary is the negative transpose of the optimal primal dictionary:

$$\begin{array}{rcrcrcrcr} -\zeta & = & -7 & - & 4y_1 & - & 3z_1 & - & 2z_2 \\ y_2 & = & \frac{1}{2} & + & y_1 & + & \frac{1}{2}z_1 & & \\ y_3 & = & \frac{5}{2} & & & + & \frac{1}{2}z_1 & + & z_2 \end{array}$$

Here variables have been replaced with their complementary counterparts. We see that the optimal objective value is $\zeta = 7$, and we obtain the optimal dual solution $y = (0, 1/2, 5/2)$. This solution is also unique.

d) What is the optimal solution if the objective in (1) is changed to $6x_1 - 3x_2$? Is the optimal solution unique?
Solution: We rewrite the objective as

$$6x_1 - 3x_2 = 6\left(3 - \frac{1}{2}w_2 - \frac{1}{2}w_3\right) - 3(2 - w_3) = 12 - 3w_2.$$

This means that the same pivots as in **b)** will give the dictionary

$$\begin{array}{rcrcrcr} \eta & = & 12 & - & 3w_2 & & \\ w_1 & = & 4 & - & w_2 & & \\ x_1 & = & 3 & - & \frac{1}{2}w_2 & - & \frac{1}{2}w_3 \\ x_2 & = & 2 & & & - & w_3 \end{array}$$

We see that the dictionary is still optimal, but the solution is not unique since w_3 can be increased from 0 to 2 without violating the constraints. The optimal solutions are thus the line segment $(3 - w_3/2, 2 - w_3)$ for $0 \leq w_3 \leq 2$, i.e., the line segment from $(3, 2)$ to $(2, 0)$.

Problem 2 (Convexity).

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a) Let $f : [a, b] \to \mathbb{R}$ be a convex function. Show that

$$\max\{f(x) : x \in [a, b]\} = \max\{f(a), f(b)\}$$

In other words, show that a convex function defined on an interval on the real line achieves its maximum in one of the end points of that interval.
Solution: Any $x \in [a, b]$ can be written on the form $(1-\lambda)a + \lambda b$ for some $0 \le \lambda \le 1$. From convexity of f it now follows that

$$\begin{aligned} f(x) &= f((1-\lambda)a + \lambda b) \le (1-\lambda)f(a) + \lambda f(b) \\ &\le (1-\lambda)\max\{f(a), f(b)\} + \lambda \max\{f(a), f(b)\} = \max\{f(a), f(b)\}, \end{aligned}$$

so that $\max\{f(x) : x \in [a, b]\} \le \max\{f(a), f(b)\}$. Since clearly also $\max\{f(x) : x \in [a, b]\} \ge \max\{f(a), f(b)\}$ the result follows.

b) Let $C \subseteq \mathbb{R}^n$ be a convex set and consider the function d_C defined by $d_C(x) = \inf\{\|x - c\| : c \in C\}$ (i.e., the smallest distance from x to C). Show that d_C is a convex function.
Hint: For points x, y, the point $(1-\lambda)x_1 + \lambda y_1$ can be useful here, where x_1 and y_1 are points in C near to achieving the (minimal) distances from x and y to C.
Solution: Let x, y be given. For any $\epsilon \ge 0$ we can find $x_1, y_1 \in C$ so that $\|x - x_1\| \le d_C(x) + \epsilon$, $\|y - y_1\| \le d_C(y) + \epsilon$. Since C is convex, $(1-\lambda)x_1 + \lambda y_1 \in C$, and we have that

$$\begin{aligned} d_C((1-\lambda)x + \lambda y) &= \inf\{\|(1-\lambda)x + \lambda y - c\| : c \in C\} \\ &\le \|(1-\lambda)x + \lambda y - ((1-\lambda)x_1 + \lambda y_1)\| \\ &\le (1-\lambda)\|x - x_1\| + \lambda\|y - y_1\| \\ &\le (1-\lambda)(d_C(x) + \epsilon) + \lambda(d_C(y) + \epsilon) \\ &= (1-\lambda)d_C(x) + \lambda d_C(y) + \epsilon. \end{aligned}$$

In the first inequality here we inserted $c = (1-\lambda)x_1 + \lambda y_1$, which is in C because of convexity. Since this applies for all ϵ it follows that $d_C((1-\lambda)x + \lambda y) \le (1-\lambda)d_C(x) + \lambda d_C(y)$ as well, so that d_C is convex.

Problem 3 (Game theory). Consider the matrix game with payoff matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & -1 \\ -1 & -2 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

In the following x denotes a (randomized) strategy for the column player, y a (randomized) strategy for the row player.

a) This question has two parts:

(i): Assume that the row player chooses strategy $y^* = (1/2, 1/2, 0, 0)$ (i.e., he chooses the first and second items with equal probability, and never the two others). What is the optimal strategy x for the column player (in order to maximize payoffs to himself)? What is the expected payoff?

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Solution: We need to solve

$$\begin{aligned}
&\max_{x\geq 0,\mathbf{1}^T x=1} \begin{pmatrix} 1/2 & 1/2 & 0 & 0\end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & -1 \\ -1 & -2 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} x \\
&= \max_{x\geq 0,\mathbf{1}^T x=1} \begin{pmatrix} -1/2 & -1 & 1/2 & 0\end{pmatrix} x \\
&= \max_{x\geq 0,\mathbf{1}^T x=1} -x_1/2 - x_2 + x_3/2 = 1/2.
\end{aligned}$$

The maximum is clearly attained for $x = (0, 0, 1, 0)$ (i.e., the column player should always choose the third item), and the expected payoff of $1/2$.

(ii): Assume that the column player chooses strategy $x^* = (0, 0, 1/2, 1/2)$ (i.e., he chooses the third and fourth items with equal probability, and never the two others). What is the optimal strategy y for the row player (in order to minimize payoffs from himself)? What is the expected payoff?

Solution: We need to solve

$$\begin{aligned}
&\min_{y\geq 0,\mathbf{1}^T y=1} y^T \begin{pmatrix} 0 & 0 & 1 & -1 \\ -1 & -2 & 0 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix} \\
&= \min_{y\geq 0,\mathbf{1}^T y=1} y^T \begin{pmatrix} 0 \\ 1/2 \\ 0 \\ -1/2 \end{pmatrix} \\
&= \min_{y\geq 0,\mathbf{1}^T y=1} (y_2 - y_4)/2 = -1/2.
\end{aligned}$$

The minimum is clearly attained for $y = (0, 0, 0, 1)$ (i.e., the row player should always choose the fourth item), with an expected payoff of $-1/2$ (i.e. the payment goes from the column player).

b) Let us instead consider the strategy $y^* = (1/3, 1/3, 0, 1/3)$ for the row player, and the strategy $x^* = (1/3, 0, 1/3, 1/3)$ for the column player. Are these strategies mutually optimal? If so, what is the value of the game?

Solution: We have that

$$Ax^* = \begin{pmatrix} 0 \\ 0 \\ 2/3 \\ 0 \end{pmatrix} \qquad (y^*)^T A = \begin{pmatrix} 0 & -2/3 & 0 & 0 \end{pmatrix}.$$

From this it follows that

$$\begin{aligned}
\max_{x\geq 0,\mathbf{1}^T x=1} (y^*)^T Ax &= \max_{x\geq 0,\mathbf{1}^T x=1} -2x_2/3 = 0 \\
\min_{y\geq 0,\mathbf{1}^T y=1} y^T Ax^* &= \min_{y\geq 0,\mathbf{1}^T y=1} 2y_3/3 = 0
\end{aligned}$$

Since the two values we obtained are equal, the two strategies are mutually optimal. The value of the game is 0, so that the game is fair.

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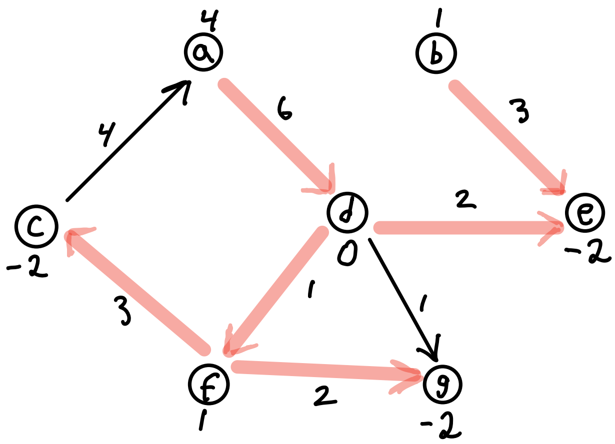

Figure 1: Flow problem for Problem 4.

Problem 4 (Network flow). Consider the minimum cost network flow problem based on the directed graph shown in Figure 1.

The number associated with each directed edge is the cost per unit flow, and the number associated with each node is the supply at that node.

a) Let T_1 be the spanning tree consisting of the edges (a,d), (d,e), (b,e), (d,f), (f,g), and (f,c) (indicated in bold in the figure). Compute the tree solution corresponding to T_1.

Solution: We apply the flow balance equations at the nodes, starting with the leafs:

- Flow balance at a gives $x_{ad} = 4$.
- Flow balance at b gives $x_{be} = 1$.
- Flow balance at c gives $-x_{fc} = -2$, so that $x_{fc} = 2$.
- Flow balance at g gives $-x_{fg} = -2$, so that $x_{fg} = 2$.
- Flow balance at e gives $-x_{de} - x_{be} = -2$, so that $x_{de} = 2 - x_{be} = 2 - 1 = 1$.
- Flow balance at d gives $x_{de} + x_{df} - x_{ad} = 0$, so that $x_{df} = x_{ad} - x_{de} = 4 - 1 = 3$.

We also have that $x_{ca} = x_{dg} = 0$, since (c,a) and (d,g) are outside T_1. The tree solution we found is (primal) feasible.

b) Use the network simplex method to find an optimal solution and optimal value for the flow problem.

Solution: Let a be the root node, so that $y_a = 0$. We first find the dual variables. We apply that $y_j - y_i = c_{ij}$ for each $(i,j) \in T_1$:

- $(i,j) = (a,d)$: $y_d - y_a = 6$, so that $y_d = 6$.
- $(i,j) = (d,e)$: $y_e - y_d = 2$, so that $y_e = 8$.
- $(i,j) = (b,e)$: $y_e - y_b = 3$, so that $y_b = 5$.

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- $(i, j) = (d, f)$: $y_f - y_d = 1$, so that $y_f = 7$.
- $(i, j) = (f, c)$: $y_c - y_f = 3$, so that $y_c = 10$.
- $(i, j) = (f, g)$: $y_g - y_f = 2$, so that $y_g = 9$.

For the dual slack variables we apply that $y_j - y_i + z_{ij} = c_{ij}$ for each $(i, j) \notin T_1$:

- $(i, j) = (c, a)$: $y_a - y_c + z_{ca} = c_{ca}$, so that $0 - 10 + z_{ca} = 4$, so that $z_{ca} = 14$.
- $(i, j) = (d, g)$: $y_g - y_d + z_{dg} = c_{dg}$, so that $9 - 6 + z_{dg} = 1$, so that $z_{dg} = -2$.

Since $z_{dg} < 0$ we do not have dual feasibility, and we thus let x_{dg} enter the basis, and increase it by ϵ. This gives a loop involving the three edges (d, f), (f, g), and (d, g). Due to the directions of the edges and flow balance, the flow is changed as follows:

$$
\begin{aligned}
\tilde{x}_{dg} &= \epsilon \\
\tilde{x}_{df} &= x_{df} - \epsilon = 3 - \epsilon \\
\tilde{x}_{fg} &= x_{fg} - \epsilon = 2 - \epsilon
\end{aligned}
$$

We see that $\tilde{x}_{fg}$ becomes zero first, and that this occurs for $\epsilon = 2$. x_{fg} thus leaves the basis. The new spanning tree is thus

$$T_2 = \{(a, d), (d, e), (b, e), (d, f), (d, g), (f, c)\}.$$

The only changes for the new tree solution are $x_{dg} = 2$, $x_{df} = 1$, $x_{fg} = 0$ (the flow can only change in the cycle introduced by the entering arc).

y_g is the only dual variable that changes (when the leaving arc is removed, the tree has two disconnected components, and only dual variables for the part not containing the root node can change), and we get $y_g - y_d = c_{dg}$, so that $y_g = 6 + 1 = 7$.

The dual slack z_{ca} does not change (since the dual variables of its terminal nodes did not change). Only z_{fg} can change, and we get $y_g - y_f + z_{fg} = c_{fg}$, so that $7 - 7 + z_{fg} = 2$, so that $z_{fg} = 2$. We thus have dual feasibility, and hence optimality. The optimal objective value is

$$
\begin{aligned}
&c_{ad}x_{ad} + c_{de}x_{de} + c_{be}x_{be} + c_{df}x_{df} + c_{dg}x_{dg} + c_{fc}x_{fc} \\
&= 6 \cdot 4 + 2 \cdot 1 + 3 \cdot 1 + 1 \cdot 1 + 1 \cdot 2 + 3 \cdot 2 \\
&= 24 + 2 + 3 + 1 + 2 + 6 = 38.
\end{aligned}
$$

Good luck!