UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3100 — Linear optimization.

Day of examination: Monday 12. June 2023.

Examination hours: 15:00 – 19:00.

This problem set consists of 2 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 (Simplex method). We will consider the following linear programming problem.

$$
\begin{array}{lccccccc}
\max & x_1 & + & 2x_2 & + & 3x_3 & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 2 \\
 & x_1 & & & & & \leq & 1 \\
 & & & x_2 & & & \leq & 1 \\
 & & & & & x_3 & \leq & 1 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
\qquad (1)
$$

a) Write down the dictionary corresponding to (1), and solve the problem using the simplex method and the largest coefficient rule. Write also down the optimal value and optimal solution. Is the optimal solution unique?

b) Write down the dual problem and the optimal dual dictionary. What is the optimal solution to the dual problem? Is this solution unique?

c) What is the optimal solution if the objective in (1) is changed to $x_1 + x_2 + 3x_3$? Is the optimal solution unique?

d) How is an extreme point of a convex set defined? What are the extreme points of the feasible region of the problem (1)?

Problem 2 (Convexity).

a) Let f be an increasing and convex function, and let g be another convex function. Show that $h(x) = f(g(x))$ is a convex function.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, and let $S = \{(x, y) : f(x) \leq y\}$. Prove that S is a convex set.

Problem 3 (Game theory). Consider the matrix game with payoff

(Continued on page 2.)

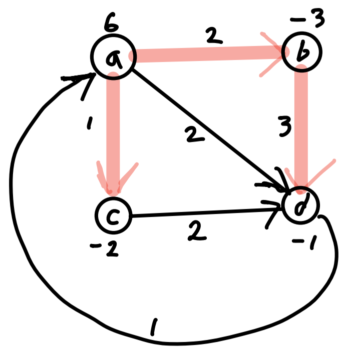

Figure 1: Flow problem for Problem 4.

matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}.$$

a) Does the matrix A have a saddle point? What does this say about the possibility of having pure strategies which are optimal?

b) This question has three parts.
(i) Assume that the row player chooses strategy $y^* = (2/3, 0, 1/3)$. Show that the expected payoff $(y^*)^T A\mathbf{x}$ is the same, regardless of the column player's strategy x. What is the expected payoff?
(ii) Assume that the column player chooses the strategy $x^* = (1/3, 1/3, 1/3)$. Show that the expected payoff $y^T A x^*$ is the same regardless of the row player's strategy y. What is the expected payoff?
(iii) What does the minimax theorem for matrix games say? Are the two strategies x^* and y^* mutually optimal? If so, what is the value of the game? Is the game fair?

Problem 4 (Network flow). Consider the minimum cost network flow problem based on the directed graph shown in Figure 1.

The number associated with each directed edge is the cost per unit flow, and the number associated with each node is the supply at that node.

a) Let T_1 be the spanning tree consisting of the edges (a, b), (a, c), and (b, d) (indicated in bold in the figure). Compute the tree solution corresponding to T_1.

b) Use the network simplex method to find an optimal solution and optimal value for the flow problem.

Good luck!