UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT3100 — Linear optimization.

Day of examination: Monday 12. June 2023.

Examination hours: 15:00 – 19:00.

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1 (Simplex method). We will consider the following linear programming problem.

$$
\begin{array}{lrcrcrcl}
\max & x_1 & + & 2x_2 & + & 3x_3 & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 2 \\
 & x_1 & & & & & \leq & 1 \\
 & & & x_2 & & & \leq & 1 \\
 & & & & & x_3 & \leq & 1 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
\tag{1}
$$

a) Write down the dictionary corresponding to (1), and solve the problem using the simplex method and the largest coefficient rule. Write also down the optimal value and optimal solution. Is the optimal solution unique?
Solution: The dictionary is

$$
\begin{array}{rclcrcrcr}
\eta & = & & & x_1 & + & 2x_2 & + & 3x_3 \\
w_1 & = & 2 & - & x_1 & - & x_2 & - & x_3 \\
w_2 & = & 1 & - & x_1 & & & & \\
w_3 & = & 1 & & & - & x_2 & & \\
w_4 & = & 1 & & & & & - & x_3
\end{array}
$$

According to the largest coefficient rule, x_3 is the entering variable. The ratios are 1/2, 0, 0, and 1. The biggest is 1, so that w_4 is the leaving variable. Substituting $x_3 = 1 - w_4$ we get

$$
\begin{array}{rclcrcrcr}
\eta & = & 3 & + & x_1 & + & 2x_2 & - & 3w_4 \\
w_1 & = & 1 & - & x_1 & - & x_2 & + & w_4 \\
w_2 & = & 1 & - & x_1 & & & & \\
w_3 & = & 1 & & & - & x_2 & & \\
x_3 & = & 1 & & & & & - & w_4
\end{array}
$$

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x_2 is now the entering variable. The ratios are now 1, 0, 1, and 0. Either w_1 or w_3 can thus leave the basis. If we choose w_1 as the leaving variable we get

$$\begin{array}{rcrcrcrcr} \eta & = & 5 & - & x_1 & - & 2w_1 & - & w_4 \\ x_2 & = & 1 & - & x_1 & - & w_1 & + & w_4 \\ w_2 & = & 1 & - & x_1 & & & & \\ w_3 & = & & & x_1 & + & w_1 & - & w_4 \\ x_3 & = & 1 & & & & & - & w_4 \end{array}$$

This dictionary is optimal, with optimal value 5, and $(x_1, x_2, x_3) = (0, 1, 1)$ as the unique optimizer.

If we instead choose w_3 as the leaving variable we get

$$\begin{array}{rcrcrcrcr} \eta & = & 5 & + & x_1 & - & 2w_3 & - & 3w_4 \\ w_1 & = & & - & x_1 & + & w_3 & + & w_4 \\ w_2 & = & 1 & - & x_1 & & & & \\ x_2 & = & 1 & & & - & w_3 & & \\ x_3 & = & 1 & & & & & - & w_4 \end{array}$$

We now need another pivot, with x_1 as entering variable. The ratios are now ∞, 1, 0, and 0, so that the leaving variable is w_1. We now obtain

$$\begin{array}{rcrcrcrcr} \eta & = & 5 & - & w_1 & - & w_3 & - & 2w_4 \\ x_1 & = & & - & w_1 & + & w_3 & + & w_4 \\ w_2 & = & 1 & + & w_1 & - & w_3 & - & w_4 \\ x_2 & = & 1 & & & - & w_3 & & \\ x_3 & = & 1 & & & & & - & w_4 \end{array}$$

This dictionary is optimal, and we get the same optimal value and unique optimizer as above.

b) Write down the dual problem and the optimal dual dictionary. What is the optimal solution to the dual problem? Is this solution unique?

Solution: The dual problem is

$$\begin{array}{lrcrcrcrcl} \text{min} & 2y_1 & + & y_2 & + & y_3 & + & y_4 & & \\ \text{subject to} & y_1 & + & y_2 & & & & & \geq & 1 \\ & y_1 & & & + & y_3 & & & \geq & 2 \\ & y_1 & & & & & + & y_4 & \geq & 3 \\ & & & & & & & y_1, y_2, y_3, y_4 & \geq & 0 \end{array}$$

The optimal dual dictionary (if w_1 was chosen to leave the basis in the second pivot in a)) is the negative transpose of the optimal primal dictionary:

$$\begin{array}{rcrcrcrcrcr} -\zeta & = & -5 & - & z_2 & - & y_2 & & & - & z_3 \\ z_1 & = & 1 & + & z_2 & + & y_2 & - & y_3 & & \\ y_1 & = & 2 & + & z_2 & & & - & y_3 & & \\ y_4 & = & 1 & - & z_2 & & & + & y_3 & + & z_3 \end{array}$$

Here variables have been replaced with their complementary counterparts. We see that the optimal objective value is $\zeta = 5$, and we obtain the optimal solution $y = (2, 0, 0, 1)$. This solution is not unique, however, since we can increase y_3 and remain at the optimal value. The first constraint says that

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we can increase y_3 to 1, the second constraint says that we can increase y_3 to 2, and the third constraint says that we can increase y_3 to infinity. We can thus increase y_3 to 1, so that the general solution is $y = (2-y_3, 0, y_3, 1+y_3)$, where $0 \leq y_3 \leq 1$.

c) What is the optimal solution if the objective in (1) is changed to $x_1 + x_2 + 3x_3$? Is the optimal solution unique?
Solution: Using the optimal dictionary from a) we rewrite the objective as

$$x_1 + x_2 + 3x_3 = x_1 + (1 - x_1 - w_1 + w_4) + 3(1 - w_4) = 4 - w_1 - 2w_4$$

This means that the same pivots as in **a)** will give an optimal dictionary for this objective as well. The optimal value is 4, but the optimiser is not unique in this case, since the nonbasic variable x_1 is not present in the objective. One sees from the optimal primal dictionary that x_1 can be increased to 1, so that the general optimiser is $(x_1, x_2, x_3) = (x_1, 1 - x_1, 1)$ for $0 \leq x_1 \leq 1$.

d) How is an extreme point of a convex set defined? What are the extreme points of the feasible region of the problem (1)?
Solution: Denote the convex set by C. A point $x \in C$ is an extreme point if, whenever x can be written as a convex combination of two points $x^1, x^2 \in C$ (i.e., $x = (1-\lambda)x^1 + \lambda x^2$), we must have that $x = x^1 = x^2$.
If you recall exercise 16 in "a mini-introduction to convexity", you know immediately that the extreme points are the seven points with coordinates being 0 or 1, with the exception of $(1, 1, 1)$. That these seven points actually are extreme points is easy to show, since if x_i is 0 or 1, component i can't be both increased and decreased (as we would need when writing x as a convex combination of two other points) and remain feasible. We also need to prove that a point x with a non-integer component can't possibly be an extreme point. This can be split in two cases, where we assume without loss of generality that x_1 is not an integer:

1. Assume that $x_1 + x_2 + x_3 < 2$. Write
$$x = \frac{1}{2}(x_1 - \epsilon, x_2, x_3) + \frac{1}{2}(x_2 + \epsilon, x_2, x_3).$$
It is clear that for small enough ϵ the sum of the components is still < 2, and still between 0 and 1, so that the points involved are feasible. It follows that x is not an extreme point.

2. Assume that $x_1 + x_2 + x_3 = 2$. Since x_1 is non-integer, one of x_2 or x_3 must also be non-integer. Assume that x_2 is. Write
$$x = \frac{1}{2}(x_1 - \epsilon, x_2 + \epsilon, x_3) + \frac{1}{2}(x_1 + \epsilon, x_2 - \epsilon, x_3).$$
For small enough ϵ, the two points involved here will still have coordinates between 0 and 1, and the coordinates still sum to 2, so the points are feasible. It follows that x is not an extreme point.

You can also prove this by considering all possible subsets of 4 columns which give an invertible submatrix B. This will required a lot of computation, though.

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Problem 2 (Convexity).

a) Let f be an increasing and convex function, and let g be another convex function. Show that $h(x) = f(g(x))$ is a convex function.
Solution: Since g is convex, $g((1-\lambda)x + \lambda y) \leq (1-\lambda)g(x) + \lambda g(y)$. Since f is increasing we have

$$f(g((1-\lambda)x + \lambda y)) \leq f((1-\lambda)g(x) + \lambda g(y)) \leq (1-\lambda)f(g(x)) + \lambda f(g(y)),$$

where we in the second inequality used that f also is convex. It follows that

$$h((1-\lambda)x + \lambda y) \leq (1-\lambda)h(x) + \lambda h(y),$$

so that h also is convex.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, and let $S = \{(x,y) : f(x) \leq y\}$. Prove that S is a convex set.
Solution: Assume that $(x_1, y_1), (x_2, y_2) \in S$, so that $f(x_1) \leq y_1$, $f(x_2) \leq y_2$. We need to show that

$$(1-\lambda)(x_1, y_1) + \lambda(x_2, y_2) = ((1-\lambda)x_1 + \lambda x_2, (1-\lambda)y_1 + \lambda y_2) \in S$$

as well. But this is equivalent to $f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)y_1 + \lambda y_2$. But this holds because of convexity of f:

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)f(x_1) + \lambda f(x_2) \leq (1-\lambda)y_1 + \lambda y_2,$$

where we used that $f(x_1) \leq y_1$, $f(x_2) \leq y_2$.

Problem 3 (Game theory). Consider the matrix game with payoff matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix}.$$

a) Does the matrix A have a saddle point? What does this say about the possibility of having pure strategies which are optimal?
Solution: A saddle point should be the biggest in the row it is in, and the smallest in the column it is in. We go through all possible rows:

- If the saddle point is in the first row, it must be the $(1,2)$-entry (the 2). But this is not the smallest in that column, so this possibility can be discarded.
- If the saddle point is in the second row, it must be the $(2,1)$-entry (the 3). But this is not the smallest in that column, so this possibility can be discarded.
- If the saddle point is in the third row, it must be either the $(3,1)$- or $(3,3)$-entry (a 2). But these are not the smallest in the respective columns, so this possibility can also be discarded.

Thus, there does not exist a saddle point. This means that there are no pure minmax/maxmin strategies, i.e., no optimal pure strategies (where one always bets on a given item) exist.

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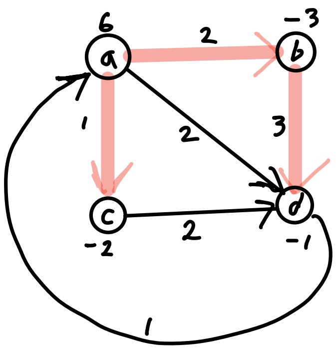

Figure 1: Flow problem for Problem 4.

b) This question has three parts.
(i) Assume that the row player chooses strategy $y^* = (2/3, 0, 1/3)$. Show that the expected payoff $(y^*)^T A\mathbf{x}$ is the same, regardless of the column player's strategy x. What is the expected payoff?
(ii) Assume that the column player chooses the strategy $x^* = (1/3, 1/3, 1/3)$. Show that the expected payoff $y^T Ax^*$ is the same regardless of the row player's strategy y. What is the expected payoff?
(iii) What does the minimax theorem for matrix games say? Are the two strategies x^* and y^* mutually optimal? If so, what is the value of the game? Is the game fair?
Solution: We have that

$$Ax^* = \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix} \qquad (y^*)^T A = \begin{pmatrix} 4/3 & 4/3 & 4/3 \end{pmatrix}.$$

From this it follows that (since x and y are stochastic)

$$\begin{aligned} (y^*)^T Ax &= \frac{4}{3}(x_1 + x_2 + x_3) = \frac{4}{3} \\ y^T Ax^* &= \frac{4}{3}(y_1 + y_2 + y_3) = \frac{4}{3} \end{aligned}$$

so that the expected payoffs are also $4/3$, for any x and y. Thus $\min_y y^T Ax^* = \max_x (y^*)^T Ax = 4/3$, where we maximise/minimise over all stochastic vectors.

The minimax theorem for matrix games says that there exist stochastic vectors x^*, y^* so that $\min_y y^T Ax^* = \max_x (y^*)^T Ax$ (where we again maximise/minimise over stochastic vectors). We have also learnt that any x^*, y^* satisfying this are mutually optimal. By the minimax theorem, x^* and y^* as defined above are thus mutually optimal (the common value is $4/3$), and the value of the game is $4/3$. Since this in nonzero, the game is not fair.

Problem 4 (Network flow). Consider the minimum cost network flow problem based on the directed graph shown in Figure 1.

The number associated with each directed edge is the cost per unit flow, and the number associated with each node is the supply at that node.

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a) Let T_1 be the spanning tree consisting of the edges (a, b), (a, c), and (b, d) (indicated in bold in the figure). Compute the tree solution corresponding to T_1.
Solution: We apply the flow balance equations at the nodes, starting with the the two leafs:

- Flow balance at c gives $-x_{ac} = -2$, so that $x_{ac} = 2$.
- Flow balance at d gives $-x_{bd} = -1$, so that $x_{bd} = 1$.
- Flow balance at a gives $x_{ab} + x_{ac} = 6$, so that $x_{ab} = 6 - x_{ac} = 4$.

We also have that $x_{cd} = x_{ad} = x_{da} = 0$, since these are outside T_1. The tree solution we found is (primal) feasible.

b) Use the network simplex method to find an optimal solution and optimal value for the flow problem.
Solution: Let a be the root node, so that $y_a = 0$. We first find the dual variables. We apply that $y_j - y_i = c_{ij}$ for each $(i, j) \in T_1$:

- $(i, j) = (a, c)$: $y_c - y_a = 1$, so that $y_c = 1$.
- $(i, j) = (a, b)$: $y_b - y_a = 2$, so that $y_b = 2$.
- $(i, j) = (b, d)$: $y_d - y_b = 3$, so that $y_d = y_b + 3 = 5$.

For the dual slack variables we apply that $y_j - y_i + z_{ij} = c_{ij}$ for each $(i, j) \notin T_1$:

- $(i, j) = (c, d)$: $y_d - y_c + z_{cd} = c_{cd}$, so that $5 - 1 + z_{cd} = 2$, so that $z_{cd} = -2$.
- $(i, j) = (a, d)$: $y_d - y_a + z_{ad} = c_{ad}$, so that $5 - 0 + z_{ad} = 2$, so that $z_{ad} = -3$.
- $(i, j) = (d, a)$: $y_a - y_d + z_{da} = c_{da}$, so that $0 - 5 + z_{da} = 1$, so that $z_{da} = 6$.

Since some of these are negative we do not have dual feasibility. According to the largest coefficient rule we should let x_{ad} enter, and increase it by ϵ. This gives a loop involving the three edges (a, b), (b, d), and (a, d). Due to the directions of the edges and flow balance, the flow is changed as follows:

$$\begin{aligned}\tilde{x}_{ab} &= x_{ab} - \epsilon = 4 - \epsilon \\ \tilde{x}_{bd} &= x_{bd} - \epsilon = 1 - \epsilon \\ \tilde{x}_{ad} &= \epsilon\end{aligned}$$

We see that $\tilde{x}_{bd}$ becomes zero first, and that this occurs for $\epsilon = 1$. x_{bd} thus leaves the basis. The new spanning tree is thus

$$T_2 = \{(a, b), (a, c), (a, d)\}.$$

The only changes for the new tree solution are $x_{ab} = 3$, $x_{ad} = 1$, $x_{bd} = 0$ (the flow can only change in the cycle introduced by the entering arc. In particular, $x_{ac} = 2$ still).

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y_d is the only dual variable that changes (when the leaving arc is removed, the tree has two disconnected components, and only dual variables for the part not containing the root node can change), and we get $y_d - y_a = c_{ad}$, so that $y_d = 0 + 2 = 2$.
The dual slacks for edges not in T_2 can be computed as follows:

- $(i, j) = (b, d)$: $y_d - y_b + z_{bd} = c_{bd}$, so that $2 - 2 + z_{bd} = 3$, so that $z_{bd} = 3$.
- $(i, j) = (c, d)$: $y_d - y_c + z_{cd} = c_{cd}$, so that $2 - 1 + z_{cd} = 2$, so that $z_{cd} = 1$.
- $(i, j) = (d, a)$: $y_a - y_d + z_{da} = c_{da}$, so that $0 - 2 + z_{da} = 1$, so that $z_{da} = 3$.

We thus have dual feasibility, and hence optimality. The optimal objective value is

$$\begin{aligned} &c_{ab}x_{ab} + c_{ac}x_{ac} + c_{ad}x_{ad} \\ &= 2 \cdot 3 + 1 \cdot 2 + 2 \cdot 1 \\ &= 6 + 2 + 2 = 10. \end{aligned}$$

Good luck!