UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 3370 — Linear optimization

Day of examination: May 31., 2011

Examination hours: 09.00 – 13.00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same weight.

Problem 1

Consider the LP problem:

$$\begin{array}{ll} \max & x_1 - 2x_2 + x_3 \\ \text{subject to} & \\ & x_1 + 2x_2 + x_3 \leq 12 \\ & 2x_1 + x_2 - x_3 \leq 6 \\ & -x_1 + 3x_2 \leq 9 \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{1}$$

1a

Solve problem (1) using the simplex algorithm with initial point $x = (0, 0, 0)$. Find an optimal solution, including values on the slack variables, and the optimal value.

Solution:

$$\begin{array}{rcrcrcrcr} \zeta & = & 0 & + & x_1 & - & 2x_2 & + & x_3 \\ \hline w_1 & = & 12 & - & x_1 & - & 2x_2 & - & x_3 \\ w_2 & = & 6 & - & 2x_1 & - & x_2 & + & x_3 \\ w_3 & = & 9 & + & x_1 & - & 3x_2 & & \end{array}$$

Pivot: x_3 in and w_1 out:

$$\begin{array}{rcrcrcrcr} \zeta & = & 12 & & & - & 4x_2 & - & w_1 \\ \hline x_3 & = & 12 & - & x_1 & - & 2x_2 & - & w_1 \\ w_2 & = & 18 & - & 3x_1 & - & 3x_2 & - & w_1 \\ w_3 & = & 9 & + & x_1 & - & 3x_2 & & \end{array}$$

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Optimal solution: $x_1 = x_2 = 0$, $x_3 = 12$ *and* $w_1 = 0$, $w_2 = 18$, $w_3 = 9$.*); optimal value is 12.*

1b

Find the dual problem of (1). Moreover, find an optimal solution of the dual, including dual slacks, preferable without any computations.

Solution: The dual is

$$\begin{array}{lrcrcrcr} \min & 12y_1 & + & 6y_2 & + & 9y_3 & & \\ \text{subject to} & & & & & & & \\ & y_1 & + & 2y_2 & - & y_3 & \geq & 1 \\ & 2y_1 & + & y_2 & + & 3y_3 & \geq & -2 \\ & y_1 & - & y_2 & & & \geq & 1 \\ & & & & y_1, y_2, y_3 \geq 0. & & & \end{array}$$

From the optimal dictionary in the previous question we see that an optimal solution in the dual is $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ *and dual slacks* $z_1 = 0$, $z_2 = 4$, $z_3 = 0$.

Let $a \leq 0$ be a parameter (number) and define the function $f_a : \mathbb{R}^3 \to \mathbb{R}$ by $f_a(x_1, x_2, x_3) = x_1 + ax_2 + x_3$. Let (P) denote the problem obtained from the LP problem (1) by replacing the objective function by $f_a(x_1, x_2, x_3)$, but using the same constraints.

1c

Find an x^* which is optimal in (P) for all $a \leq 0$, and show that it is optimal.

Solution: $x^* = (0, 0, 12)$ *(found in a)) is optimal in (P), and the optimal value is 12. Proof: The solution* x^* *is feasible (shown before), and* $f_a(x^*) = 12$. *Moreover, by adding the constraint* $x_1 + 2x_2 + x_3 \leq 12$ *and* $a - 2$ *(which is negative) times the constraint* $x_2 \geq 0$ *we get* $x_1 + 2x_2 + x_3 + (a - 2)x_2 \leq 12 + 0 = 12$. *So*

$$f_a(x) = x_1 + ax_2 + x_3 \leq 12$$

Thus each feasible solution in (P) must satisfy this constraint. This implies that x^* *is optimal.*

Alternative: use simplex algorithm with the same pivots as in a) and note that the same solution is optimal (using that $a \leq 0$*).*

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Problem 2

Consider the LP problem:

$$\begin{array}{llcccccccc} \max & -x_1 & - & x_2 & - & x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & 2x_2 & + & x_3 & \le & 12 \\ & 2x_1 & + & x_2 & - & x_3 & \le & -6 \\ & & & x_1, x_2, x_3 \ge 0. & & & & \end{array} \tag{2}$$

2a

Use the dual simplex method to find a feasible solution in (2). (If you don't remember this method, you may also get some (but not full) score using another method.) Is there a basic feasible solution in (2) where x_3 is not a basic variable (i.e., x_3 is not in the basis)? Give reasons for your answer.

Solution:

$$\begin{array}{ccrcrcrcr} \zeta & = & 0 & - & x_1 & - & x_2 & - & x_3 \\ \hline w_1 & = & 12 & - & x_1 & - & 2x_2 & - & x_3 \\ w_2 & = & -6 & - & 2x_1 & - & x_2 & + & x_3 \end{array}$$

Pivot: w_2 out of basis and x_3 in which gives:

$$\begin{array}{ccrcrcrcr} \zeta & = & -6 & - & 3x_1 & - & 2x_2 & - & w_2 \\ \hline w_1 & = & 6 & - & 3x_1 & - & 3x_2 & - & w_2 \\ x_3 & = & 6 & + & 2x_1 & + & x_2 & + & w_2 \end{array}$$

So $x_1 = x_2 = 0$, $x_3 = 6$ (and $w_1 = 6$, $w_2 = 0$) is a feasible solution.

Second question: No. Because if x_3 is nonbasic variable it is zero, and then the the second equation becomes

$$-6 = 2x_1 + x_2 - x_3 + w_2 = 2x_1 + x_2 + w_2$$

Since all variables are nonnegative, $2x_1 + x_2 + w_2 \ge 0$ and it cannot be equal to -6. So, x_3 must be in the basis for each basic feasible solution.

Problem 3

Let $a \in \mathbb{R}^n$ where $\|a\| = \sqrt{a^T a} = 1$, and let $b_1, b_2 \in \mathbb{R}$ with $b_1 < b_2$. Define $H_1 = \{x \in \mathbb{R}^n : a^T x = b_1\}$ and $H_2 = \{x \in \mathbb{R}^n : a^T x = b_2\}$.

3a

Determine the convex hull of $H_1 \cup H_2$ (with proof).

Solution: Define $C = \{x \in \mathbb{R}^n : b_1 \le a^T x \le b_2\}$. We claim that $\text{conv}(H_1 \cup H_2) = C$. *First, C is a polyhedron and therefore convex, and it*

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contains $H_1 \cup H_2$. *This implies that* $\text{conv}(H_1 \cup H_2) \subseteq C$. *Next, if* $x_0 \in C$ *then* $b_1 \leq a^T x_0 \leq b_2$. *Consider* $x^1 = x_0 - (a^T x_0 - b_1)a$ *and* $x^2 = x_0 + (b_2 - a^T x_0)a$. *Then* $x^1 \in H_1$, $x^2 \in H_2$ *(by computing the inner products) and* x_0 *is clearly on the line segment between* x^1 *and* x^2. *Therefore* $x_0 \in \text{conv}(H_1 \cup H_2)$. *This proves that* $\text{conv}(H_1 \cup H_2) \supseteq C$, *and therefore these two sets are equal.*

Problem 4

Consider the LP problem

$$\begin{array}{ll} \max & c^T x \\ \text{subject to} & \\ & Ax = b \\ & O \leq x \leq h \end{array} \tag{3}$$

Here $c, h \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and the $m \times n$ matrix A are given, and O denotes the zero vector.

4a

Find the dual of problem (3).

Solution: Problem (2) may be written

$$\max\{c^T x : \begin{bmatrix} A \\ -A \\ I \end{bmatrix} x \leq \begin{bmatrix} b \\ -b \\ h \end{bmatrix}, x \geq O\}$$

so the dual is (with variables $y_1, y_2 \in \mathbb{R}^m$ *and* $z \in \mathbb{R}^n$*):*

$$\min\{b^T y_1 - b^T y_2 + h^T z : \begin{bmatrix} A^T & -A^T & I \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ z \end{bmatrix} \geq c,\ y_1, y_2, z \geq O\}$$

or

$$\min\{b^T(y_1 - y_2) + h^T z : A^T(y_1 - y_2) + z \geq c,\ y_1, y_2, z \geq O\}$$

By replacing $y_1 - y_2$ *by* y, *the dual is equivalent to*

$$\min\{b^T y + h^T z : A^T y + z \geq c,\ y \text{ free},\ z \geq O\}$$

Consider the linear system

$$\begin{array}{l} \sum_{j=1}^n x_j \leq 1 \\ O \leq x_j \leq h_j \quad (j \leq n) \end{array} \tag{4}$$

where each h_j is (strictly) positive. (Note: there are only inequalities.)

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4b

Use Fourier-Motzkin elimination to eliminate x_1 in (4). Then go on and eliminate x_2, x_3 etc.; explain how x_k depends on $x_{k+1}, \ldots, x_n$ in a general solution of (4).

Solution: Eliminate x_1*: From* $0 \leq x_j$ *and* $x_j \leq h_j$ *and* $\sum_{j=1}^n x_j \leq 1$ *we get* $0 \leq x_1 \leq \min\{h_1, 1 - \sum_{j=2}^n x_j\}$ *and the new (projected) system*

$$\begin{array}{l} \sum_{j=2}^n x_j \leq 1 \\ O \leq x_j \leq h_j \quad (2 \leq j \leq n) \end{array}$$

We repeat the process, and it is easy to show by induction that when x_{k-1} *is eliminated (for some* k*) we get* $0 \leq x_k \leq \min\{h_k, 1 - \sum_{j=k+1}^n x_j\}$ *and the new (projected) system*

$$\begin{array}{l} \sum_{j=k+1}^n x_j \leq 1 \\ O \leq x_j \leq h_j \qquad (k+1 \leq j \leq n) \end{array}$$

Problem 5

Consider the following minimum cost network flow problem. Define the directed graph $D = (V, E)$ where $V = \{v_1, v_2, \ldots, v_5\}$ (the nodes) and E (the edges=arcs) consists of (v_i, v_{i+1}) for $1 \leq i \leq 4$ and the edge (v_2, v_4). So D has 5 edges. Define the supply vector b by $b_{v_1} = 1$, $b_{v_5} = -1$ and $b_{v_i} = 0$ for $i = 2, 3, 4$. Finally, define the cost $c_e = 1$ for every edge e.

5a

Draw the graph. Find all spanning trees in D (for each, give its edges). Let T_1 be the spanning tree which does not contain (v_3, v_4), and compute the corresponding tree solution x^*.

Solution: (Draw e.g. along a line.) There are 3 Spanning trees: $E \setminus \{(v_3, v_4)\}$, $E \setminus \{(v_2, v_3)\}$ *and* $E \setminus \{(v_2, v_4)\}$*. Computing* x^* *by leaf elimination:* $x_{v_3 v_4} = 0$ *and* $x_{v_2 v_3} = 0$ *while* $x_e = 1$ *for all other edges* e*. (The solution corresponds to a shortest* $v_1 v_5$*-path.)*

5b

Compute the dual solution (y, z) corresponding to the spanning tree T_1 above: let here $y_{v_1} = 0$ (so v_1 is the root). Explain why x^* above is an optimal solution of the minimum cost network flow problem.

Solution: Computing y*: use* $y_v - y_u = c_{uv}$ *for each edge in the tree. This gives:* $y_{v_1} = 0$, $y_{v_2} = 1$, $y_{v_3} = y_{v_4} = 2$ *and* $y_{v_5} = 3$*. Computing* z*: use* $z_{uv} = y_u + c_{uv} - y_v$ *for each edge* (u, v) *not in the tree (the other* z*'s are zero):*

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This gives $z_{v_3v_4} = 2 + 1 - 2 = 1 \geq 0$*. So* $z \geq O$ *and the present basis is optimal (by the network simplex algorithm); so* x^* *is optimal.*

Let A be the node-arc (node-edge) incidence matrix of the graph D above.

5c

Let r be the maximum rank of a square submatrix of A. Find r and a submatrix B of A such that B has rank r. Explain your answer with reference to general theory.

Solution: Since D *is connected* A *has rank equal to* $|V| - 1 = 4$ *(by Proposition in lecture notes, or Theorem 14.1 in Vanderbei). Therefore each square submatrix has rank* ≤ 4*. By Theorem 14.1 such a submatrix* B *has rank 4 if and only if its columns correspond to the edges of a spanning tree. Therefore* $r = 4$*. So if we use the spanning tree* T_1*, and order nodes/edges according to tree elimination, we get a submatrix* B *with rows corresponding to (e.g.)* v_5, v_4, v_3, v_2 *and columns corresponding to* (v_4, v_5), (v_2, v_4), (v_2, v_3), (v_1, v_2), *(in this order). Then*

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$$

which is unit upper triangular and invertible (nonsingular) so it has rank 4.