UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in INF-MAT 3370 — Linear optimization

Day of examination: June 5, 2012

Examination hours: 09.00 – 13.00

This problem set consists of 4 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same weight.

Problem 1

Consider the LP problem:

$$\begin{array}{lrcrcrcl} \max & -2x_1 & + & 4x_2 & + & 2x_3 & & \\ \text{subject to} & & & & & & & \\ & -x_1 & - & x_2 & & & \leq & 0 \\ & -3x_1 & + & x_2 & - & 2x_3 & \leq & 1 \\ & x_1 & - & x_2 & - & 3x_3 & \leq & 3 \\ & & & x_1, x_2, x_3 \geq 0. & & & & \end{array} \tag{1}$$

1a

Solve problem (1) using the simplex algorithm. Find, if possible, a feasible solution (point) with value 14 on the objective function (which is $f(x) = -2x_1 + 4x_2 + 2x_3$).

Let (P2) be the LP problem obtained from problem (1) by adding the constraint

$$x_1 + x_2 + x_3 \leq 8.$$

1b

Explain, without solving (P2) numerically, why (P2) has an optimal solution.

(Continued on page 2.)

1c

What is Bland's rule? Explain briefly its purpose and why it may not be very efficient in practice.

Consider the LP problem given by the following dictionary

$$\begin{array}{rclrrrrrr} \zeta & = & 0 & - & x_1 & - & 3x_2 & - & x_3 \\ \hline w_1 & = & -5 & + & x_1 & + & 2x_2 & - & x_3 \\ w_2 & = & -1 & + & 2x_1 & - & x_2 & + & x_3 \end{array}$$

1d

Solve the problem using the dual simplex algorithm. Find both an optimal primal solution and an optimal solution of the dual, and the optimal value.

Problem 2

Consider the matrix game given by the following 3×4 matrix A

$$A = \begin{bmatrix} 2 & 7 & 6 & 10 \\ 1 & 3 & 3 & 2 \\ 2 & 0 & 5 & 4 \end{bmatrix}.$$

2a

Find a pure minmax strategy for the row player R and a pure maxmin strategy for the column player K. Also determine the value of the game.

Consider the 3×3 matrix

$$A_x = \begin{bmatrix} x_1 & x_2 & x_3 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

which depends on the parameter vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

2b

Assume that x is nonnegative and satisfies $x_1 + x_2 + x_3 = 1$. How large can the determinant of A_x be? Find all x such that $\det A_x$ attains this maximum value.

(Continued on page 3.)

Problem 3

Consider the minimum cost network flow problem in the directed graph D shown in Figure 1. The four nodes (vertices) are u, v, w and p and the numbers along the edges are the costs. The supply/demand is given by $b_u = 4$, $b_v = -1$, $b_w = 0$, $b_p = -3$ (so u is a supply node, while v and p are demand nodes).

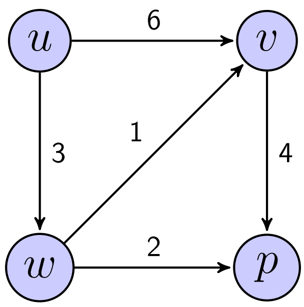

Figure 1: The directed graph D.

3a

Compute the tree solution x that corresponds to the tree T_1 with edges (u, v), (u, w) and (v, p). Can this x be obtained as the tree solution of *another* spanning tree as well? Explain your answer.

3b

Find an optimal solution (and the optimal value) of the network flow problem, and indicate your computations.

Problem 4

Let P be the set of all solutions to the linear system

$$7x_1 + x_2 + 4x_3 \leq 8$$
$$0 \leq x_1, x_2, x_3 \leq 1$$

(so each variable lies in the interval $[0, 1]$).

(Continued on page 4.)

4a

Find all extreme points of P.

Consider the set

$$K = \{x \in \mathbb{R}^n : Ax \leq b,\, x = Cy,\, y \geq O, \sum_{i=1}^{k} y_i = 1\}$$

where k, m and n are positive integers, A is an $m \times n$ matrix, $b \in \mathbb{R}^m$ and C is an $n \times k$ matrix. Here O denotes the zero vector. (So: $x \in K$ means that there exists a y such that all the conditions indicated hold.)

4b

Prove that K is a polyhedron in $\mathbb{R}^n$.

Good luck!