UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: INF-MAT 3370 — Linear optimization

Day of examination: June 5, 2012

Examination hours: 09.00 – 13.00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same qeight.

Solutions.

Problem 1

Consider the LP problem:

$$\begin{array}{lrcrcrcl}
\text{max} & -2x_1 & + & 4x_2 & + & 2x_3 & & \\
\text{subject to} & & & & & & & \\
 & -x_1 & - & x_2 & & & \leq & 0 \\
 & -3x_1 & + & x_2 & - & 2x_3 & \leq & 1 \\
 & x_1 & - & x_2 & - & 3x_3 & \leq & 3 \\
 & & & \multicolumn{3}{c}{x_1, x_2, x_3 \geq 0.} & &
\end{array} \qquad (1)$$

1a

Solve problem (1) using the simplex algorithm. Find, if possible, a feasible solution (point) with value 14 on the objective function (which is $f(x) = -2x_1 + 4x_2 + 2x_3$).

Solution:

$$\begin{array}{rcrcrcrcr}
\zeta & = & 0 & - & 2x_1 & + & 4x_2 & + & 2x_3 \\
\hline
w_1 & = & 0 & + & x_1 & + & x_2 & & \\
w_2 & = & 1 & + & 3x_1 & - & x_2 & + & 2x_3 \\
w_3 & = & 3 & - & x_1 & + & x_2 & + & 3x_3
\end{array}$$

(Continued on page 2.)

Pivot: x_2 in and w_2 out:

$$\begin{array}{cclclclcl}
\zeta & = & 4 & + & 10x_1 & - & 4w_2 & + & 10x_3 \\
\hline
w_1 & = & 1 & + & 4x_1 & - & w_2 & + & 2x_3 \\
x_2 & = & 1 & + & 3x_1 & - & w_2 & + & 2x_3 \\
w_3 & = & 4 & + & 2x_1 & - & w_2 & + & 5x_3
\end{array}$$

The problem is unbounded as we may increase x_1 towards infinity (among feasible solutions). Let $x_1 = 1$, $x_2 = 4$, $x_3 = 0$ (and $w_1 = 5$, $w_2 = 0$, $w_3 = 6$). Then $x = (x_1, x_2, x_3)$ is feasible and $f(x) = 14$.

□

Let (P2) be the LP problem obtained from problem (1) by adding the constraint

$$x_1 + x_2 + x_3 \leq 8.$$

1b

Explain, without solving (P2) numerically, why (P2) has an optimal solution.

Solution: Due to the new constraint each feasible $x = (x_1, x_2, x_3)$ in (P2) satisfies $0 \leq x_i \leq 8$ so the feasible set is bounded. Therefore (P2) is feasible (e.g. the zero vector is feasible) and not unbounded, and by the Fundamental Theorem of LP it must have an optimal solution. (Alternative solution: The conclusion also follows from the Extreme Value Theorem: the feasible set is nonempt,y closed and bounded and f is continuous, so the maximum is attained.)

□

1c

What is Bland's rule? Explain briefly its purpose and why it may not be very efficient in practice.

See Vanderbei. Good for theoretical purposes: shows that the simplex algorithm terminates with this anti-cycling rule. May be very bad in practice as, e.g., the incoming variable may have very small coefficient and give small improvement; many pivots. □

Consider the LP problem given by the following dictionary

$$\begin{array}{cclclclcl}
\zeta & = & 0 & - & x_1 & - & 3x_2 & - & x_3 \\
\hline
w_1 & = & -5 & + & x_1 & + & 2x_2 & - & x_3 \\
w_2 & = & -1 & + & 2x_1 & - & x_2 & + & x_3
\end{array}$$

(Continued on page 3.)

1d

Solve the problem using the dual simplex algorithm. Find both an optimal primal solution and an optimal solution of the dual, and the optimal value.

Solution: w_1 *leaves the basis (as* $-5 < -1$*) and the ratio for* x_1 *is* $1/1 = 1$ *and for* x_2 *the ratio is* $3/2$*, so* x_1 *enters the basis. The pivot gives*

$$\begin{array}{rcrcrcrcr} \zeta & = & -5 & - & w_1 & - & x_2 & - & 2x_3 \\ \hline x_1 & = & 5 & + & w_1 & - & 2x_2 & + & x_3 \\ w_2 & = & 9 & + & 2w_1 & - & 5x_2 & + & 3x_3 \end{array}$$

So optimal primal solution: $x_1 = 5$, $x_2 = 0$, $x_3 = 0$, $w_1 = 0$, $w_2 = 9$. *Optimal value is* -5. *Optimal dual solution:* $z_1 = 0$, $z_2 = 1$, $z_3 = 2$, $y_1 = 1$, $y_2 = 0$.

□

Problem 2

Consider the matrix game given by the following 3×4 matrix A

$$A = \begin{bmatrix} 2 & 7 & 6 & 10 \\ 1 & 3 & 3 & 2 \\ 2 & 0 & 5 & 4 \end{bmatrix}.$$

2a

Find a pure minmax strategy for the row player R and a pure maxmin strategy for the column player K. Also determine the value of the game.

Solution: The pair $(2, 3)$ *is a saddle point as* $a_{23} = 3$ *is the minimum in column 3 and the maximum in row 2. By a theorem (see lecture/book) this means that (row) 2 is a pure minmax strategy for player R and (column) 3 is a pure maxmin strategy for player K. The value of the game is* $V = a_{23} = 3$.

□

Consider the 3×3 matrix

$$A_x = \begin{bmatrix} x_1 & x_2 & x_3 \\ 3 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

which depends on the parameter vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

(Continued on page 4.)

2b

Assume that x is nonnegative and satisfies $x_1 + x_2 + x_3 = 1$. How large can the determinant of A_x be? Find all x such that $\det A_x$ attains this maximum value.

Solution: $f(x) = \det A_x = x_1(1-0) - x_2(3-2) + x_3(0-2) = x_1 - x_2 - 2x_3$. *So we get the LP*

$$\begin{array}{llcccccccl} \max & x_1 & - & x_2 & - & 2x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2 & + & x_3 & = & 1 \\ & & & & x_1, x_2, x_3 \geq 0. & & & \end{array}$$

The unique optimal solution is $x = (1, 0, 0)$ *(solve by simplex method, or just check the three basic solutions). The maximum value of* $\det A_x$ *is therefore 1.*

□

Problem 3

Consider the minimum cost network flow problem in the directed graph D shown in Figure 1. The four nodes (vertices) are u, v, w and p and the numbers along the edges are the costs. The supply/demand is given by $b_u = 4$, $b_v = -1$, $b_w = 0$, $b_p = -3$ (so u is a supply node, while v and p are demand nodes).

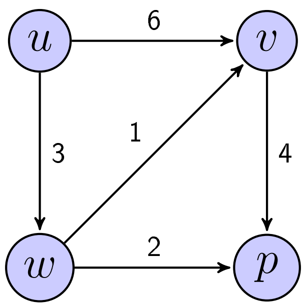

Figure 1: The directed graph D.

3a

Compute the tree solution x that corresponds to the tree T_1 with edges (u, v), (u, w) and (v, p). Can this x be obtained as the tree solution of *another* spanning tree as well? Explain your answer.

(Continued on page 5.)

Solution: Use leaf elimination, and compute (for instance in the order (v, p), (u, v) and (u, w)). This gives $x_{vp} = 3$, $x_{uv} = 4$, $x_{uw} = 0$ and $x_{wv} = x_{wp} = 0$. Next, x is also the tree solution for two other spanning trees, those obtained from T_1 by replacing (u, w) by either (w, v) or (w, p). $\square$

3b

Find an optimal solution (and the optimal value) of the network flow problem, and indicate your computations.

Solution: Start with the spanning tree T_1 and x above. Use node u as the root. We compute y using $y_j = y_i + c_{ij}$ for each edge (i, j) in T_1: $y_u = 0$, $y_w = 3$, $y_v = 6$, $y_p = 10$. Next we compute z from $z_{ij} = y_i + c_{ij} - y_j$ for each edge (i, j) (this is zero for edges in the tree). We get: $z_{wv} = -2$ and $z_{wp} = -5$. So not optimal.

Pivot: Take (w, p) into the basis. Let $x_{wp} = \epsilon$, so $x_{vp} = 3-\epsilon$, $x_{uv} = 4-\epsilon$, $x_{uw} = \epsilon$. Maximum ϵ is 3. New spanning tree is T_1 with edges (u, v), $u, w)$ and (w, p). The tree solution is now $x_{uv} = 1$, $x_{uw} = 3$, $x_{wp} = 3$ and $x_{wv} = x_{wp} = 0$. Dual solution: $y_u = 0$, $y_w = 3$, $y_v = 6$, $y_p = 5$ and $z_{wv} = -2$ and $z_{wp} = 5$. So not optimal.

Pivot: Take (w, v) into the basis. This gives $x_{uw} = 4$, $x_{wv} = 1$, $x_{wp} = 3$ and $x_{uv} = x_{vp} = 0$. Dual solution: $y_u = 0$, $y_w = 3$, $y_v = 4$, $y_p = 5$ and $z_{uv} = 2$ and $z_{vp} = 3$. So optimal. The optimal value is 19.

$\square$

Problem 4

Let P be the set of all solutions to the linear system

$$7x_1 + x_2 + 4x_3 \leq 8$$
$$0 \leq x_1, x_2, x_3 \leq 1$$

(so each variable lies in the interval $[0, 1]$).

4a

Find all extreme points of P.

Solution: See the example in Section 7 in the lecture notes on convexity. P is a polyhedron and either of the two techniques may be used. First we find extreme points that are $(0, 1)$-vectors satisfying the inequality $7x_1+x_2+4x_3 \leq 8$. This gives: $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$, $(0, 1, 1)$ (but NOT $(1, 0, 1)$). Moreover, we compute extreme points satisfying $7x_1+x_2+4x_3 = 8$ with at most one component strictly between 0 and 1. This gives $(4/7, 0, 1)$, $(1, 0, 1/4)$ and $(3/7, 1, 1)$. These are all the extreme points.

(Continued on page 6.)

□

Consider the set

$$K = \{x \in \mathbb{R}^n : Ax \le b,\, x = Cy,\, y \ge O, \sum_{i=1}^{k} y_i = 1\}$$

where k, m and n are positive integers, A is an $m \times n$ matrix, $b \in \mathbb{R}^m$ and C is an $n \times k$ matrix. Here O denotes the zero vector. (So: $x \in K$ means that there exists a y such that all the conditions indicated hold.)

4b

Prove that K is a polyhedron in $\mathbb{R}^n$.

Solution: Consider the set

$$L = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^k : x = Cy,\, y \ge O, \sum_{i=1}^{k} y_i = 1\}.$$

This is a set in $\mathbb{R}^n \times \mathbb{R}^k$ which is the solution set of a finite system of linear equations and inequalities. Therefore L is a polyhedron. We now use Fourier-Motzkin elimination on the system defining L and eliminate $y_1, y_2, \ldots, y_k$, one variable at the time. From a theorem in "A mini-introduction to convexity" we then obtain the projection of L into the space of the x-variable; call this set P_L. Moreover, the new system (from FM-elimination) is a linear system, so the projectionP_L is a polyhedron. Now

$$\begin{aligned} K &= \{x \in \mathbb{R}^n : Ax \le b,\, x = Cy,\, y \ge O, \textstyle\sum_{i=1}^{k} y_i = 1\} \\ &= \{x \in \mathbb{R}^n : x = Cy,\, y \ge O, \textstyle\sum_{i=1}^{k} y_i = 1\} \cap \{x \in \mathbb{R}^n : Ax \le b\} \\ &= P_L \cap \{x \in \mathbb{R}^n : Ax \le b\} \end{aligned}$$

But $\{x : Ax \le b\}$ is also a polyhedron, and the intersection of two polyhedra is a polyhedron (follows directly from the definition). Therefore K is a polyhedron in $\mathbb{R}^n$.

□