UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: INF-MAT 3370 — Linear optimization

Day of examination: May 29, 2013

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same weight.

Problem 1

Consider the LP problem

$$
\text{(P)} \quad \begin{array}{lrcrcrcl}
\max & x_1 & + & 2x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & 4x_2 & + & 3x_3 & \leq & 6 \\
& 3x_1 & + & x_2 & + & 2x_3 & \leq & 5 \\
& & & & & \multicolumn{3}{l}{x_1, x_2, x_3 \geq 0.}
\end{array}
$$

1a

Solve problem (P) using the simplex algorithm. Find the optimal value and an optimal solution. Write down the dual problem (D) of (P).

1b

Find an optimal solution of (D). Find *all* optimal solutions of problem (P).

1c

Give the definition of a *polyhedron* and an *extreme point* of a polyhedron. Let F be the feasible set in (P), i.e., those points $x \in \mathbb{R}^3$ that satisfy all the five constraints in (P). Decide if $x = (1, 1, 0)$ is an extreme point of F, and explain why.

(Continued on page 2.)

Problem 2

Let A and B be (real) matrices with n columns, and let b and c be vectors (of suitable sizes), and let O denote the zero vector. Consider the LP problem

$$\text{(P)} \quad \begin{array}{ll} \max & c^T x \\ \text{subject to} & \\ & Ax \leq b \\ & Bx = O \\ & x \geq O. \end{array}$$

2a

Find the dual of (P).

Problem 3

Consider the minimum cost network flow problem in the directed graph D indicated in Figure 1. There are five nodes (vertices), u, v etc. The numbers along the edges are the costs. The supply/demand is given by $b_u = 1$, $b_v = 0$, $b_w = 3$, $b_p = -1$ and $b_q = -3$ (so, for instance, u is a supply node, while q is a demand node).

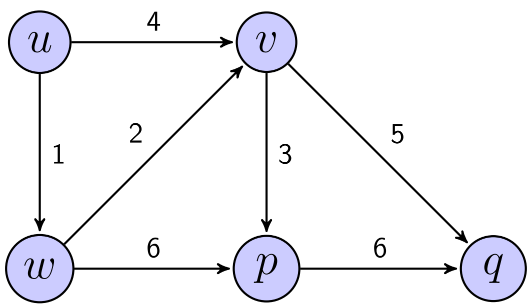

Figure 1: The directed graph D.

3a

Let T_1 consist of (all nodes and) the edges (u, w), (w, p), (w, v) and (v, q). Explain why T_1 is a spanning tree. Compute the tree solution $x = x_{T_1}$ associated with T_1.

3b

Use the network simplex algorithm to find an optimal solution, and the optimal value, of the network flow problem. Show the computations.

(Continued on page 3.)

3c

Let A be the incidence matrix of the directed graph D above (i.e., the coefficient matrix of the flow balance equations). What is the rank of A? Consider again the solution $x = x_{T_1}$ from question a). Assume that there are (feasible) flow vectors x^1 and x^2 in the network flow problem such that

$$x = 0.2\,x^1 + 0.8\,x^2$$

What can you say about x^1 and x^2? Explain.

Problem 4

Let α and β be real numbers and consider the matrix game given by the following 3×4 matrix A (which depends on α, β)

$$A = \begin{bmatrix} 3 & 8 & 7 & 11 \\ 2 & 4 & \alpha & \beta \\ 3 & 1 & 6 & 7 \end{bmatrix}.$$

4a

Determine the set H consisting of all $(\alpha, \beta) \in \mathbb{R}^2$ such that (row) 2 is a pure minmax strategy for the row player R and (column) 3 is a pure maxmin strategy for the column player K.

Problem 5

Let H be a real $m \times n$ matrix and let $c \in \mathbb{R}^m$. Assume that a vector $\bar{z} \in \mathbb{R}^n$ satisfies $H\bar{z} = c$ and $\bar{z} \geq O$ (so $\bar{z}$ is nonnegative).

5a

Prove that $Hz = c$ has a nonnegative solution z with at most m positive variables. (Hint: consider the LP problem $\min\, \{\sum_{i=1}^{m} w_i : Hz + w = c,\ z \geq O,\ w \geq O\}$.)

Consider a linear system $Ax \leq b$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

5b

State Farkas' lemma for the system $Ax \leq b$. Show that if $Ax \leq b$ is inconsistent (meaning: has no solution), then $Ax \leq b$ contains a subsystem with at most $n + 1$ inequalities which is also inconsistent.

Good luck!