UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: INF-MAT 3370 — Linear optimization

Day of examination: May 29, 2013

Examination hours: 14.30 – 18.30

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same weight.

Solutions.

Problem 1

Consider the LP problem

$$\text{(P)}\quad \begin{array}{lrcrcrcl} \text{max} & x_1 & + & 2x_2 & + & 3x_3 & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & 4x_2 & + & 3x_3 & \leq & 6 \\ & 3x_1 & + & x_2 & + & 2x_3 & \leq & 5 \\ & & & x_1, x_2, x_3 \geq 0. & & & & \end{array}$$

1a

Solve problem (P) using the simplex algorithm. Find the optimal value and an optimal solution. Write down the dual problem (D) of (P).

Solution:

$$\begin{array}{rcrcrcrcr} \zeta & = & 0 & + & x_1 & + & 2x_2 & + & 3x_3 \\ \hline x_4 & = & 6 & - & x_1 & - & 4x_2 & - & 3x_3 \\ x_5 & = & 5 & - & 3x_1 & - & x_2 & - & 2x_3 \end{array}$$

Pivot: x_3 in and x_4 out:

$$\begin{array}{rcrcrcrcr} \zeta & = & 6 & + & 0x_1 & - & 2x_2 & - & x_4 \\ \hline x_3 & = & 2 & - & (1/3)x_1 & - & (4/3)x_2 & - & (1/3)x_4 \\ x_5 & = & 1 & - & (7/3)x_1 & + & (5/3)x_2 & + & (2/3)x_4 \end{array}$$

(Continued on page 2.)

So the optimal value is 6, and an optimal solution is $x^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (0, 0, 2, 0, 1)$.

The dual (D) is

$$\begin{array}{lrcrcl} \text{min} & 6y_1 & + & 5y_2 & & \\ \text{subject to} & & & & & \\ & y_1 & + & 3y_2 & \geq & 1 \\ & 4y_1 & + & y_2 & \geq & 2 \\ & 3y_1 & + & 2y_2 & \geq & 3 \\ & & & y_1, y_2 \geq 0. & & \end{array}$$

□

1b

Find an optimal solution of (D). Find *all* optimal solutions of problem (P).

Solution: From the optimal dictionary above we see that $y_1 = 1$, $y_2 = 0$ *is an optimal solution in (D); here* y_1 *and* y_2 *corresponds to* x_4 *and* x_5 *(the slack variables in (P)), respectively.*

Any other optimal solution x *than* x^* *above must satisfy* $x_2 = x_4 = 0$, *otherwise a lower value than 6 would be obtained (as their coefficients in the objective function are strictly negative).* x_1 *has coefficient 0, so it may be increased and the objective remains at 6; the maximum value is* $x_1 = 3/7$ *and then* $x_3 = 2 - (1/3)(3/7) = 13/7$. *So another optimal basic solution is* $x' = (3/7, 0, 13/7, 0, 0)$. *(This argument just simplifies a little compared to making the full pivot where* x_1 *enters and* x_5 *leaves the basis). Then* x^* *and* x' *are the only optimal basic solutions so the set of optimal solutions is the line segment between* x^* *and* x' *(i.e., all convex combinations of these two points).*

□

1c

Give the definition of a *polyhedron* and an *extreme point* of a polyhedron. Let F be the feasible set in (P), i.e., those points $x \in \mathbb{R}^3$ that satisfy all the five constraints in (P). Decide if $x = (1, 1, 0)$ is an extreme point of F, and explain why.

Solution: A polyhedron is a set P *of the form* $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ *where* A *is a matrix and* b *is a vector. An extreme point of* P *is a point which cannot be written as a convex combination of other points in* P *(or, equivalently, as the midpoint on the line segment between two other points in* P.*)*

(Continued on page 3.)

First, $x \in F$, so it is feasible. But $x = (1/2)x^1 + (1/2)x^2$ where $x^1 = (1.1, 1, 0)$ and $x^2 = (0.9, 1, 0)$. Since $x^1, x^2 \in F$, this shows (by definition) that x is not an extreme point of F. □

Problem 2

Let A and B be (real) matrices with n columns, and let b and c be vectors (of suitable sizes), and let O denote the zero vector. Consider the LP problem

$$\text{(P)} \quad \begin{array}{ll} \max & c^T x \\ \text{subject to} & \\ & Ax \le b \\ & Bx = O \\ & x \ge O. \end{array}$$

2a

Find the dual of (P).

Solution: Problem (P) may be written

$$\max\{c^T x : Ax \le b, Bx = O,\ x \ge O\} \quad = \max\{c^T x : \begin{bmatrix} A \\ B \\ -B \end{bmatrix} x \le \begin{bmatrix} b \\ O \\ O \end{bmatrix},\ x \ge O\}.$$

So the dual is

$$\min\{b^T y_1 : A^T y_1 + B^T y_2 + (-B^T) y_3 \ge c,\ y_1, y_2, y_3 \ge O\}$$

which becomes (by letting $y_0 = y_2 - y_3$, a free variable)

$$\min\{b^T y_1 : A^T y_1 + B^T y_0 \ge c,\ y_1 \ge O,\ y_0 \text{ is free}\}$$

□

Problem 3

Consider the minimum cost network flow problem in the directed graph D indicated in Figure 1. There are five nodes (vertices), u, v etc. The numbers along the edges are the costs. The supply/demand is given by $b_u = 1$, $b_v = 0$, $b_w = 3$, $b_p = -1$ and $b_q = -3$ (so, for instance, u is a supply node, while q is a demand node).

(Continued on page 4.)

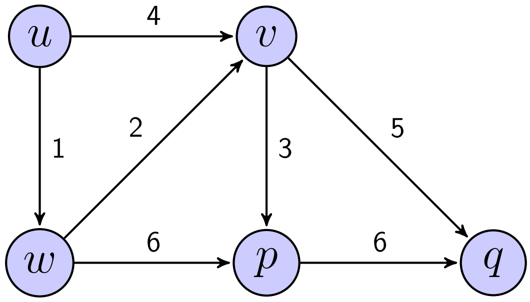

Figure 1: The directed graph D.

3a

Let T_1 consist of (all nodes and) the edges (u,w), (w,p), (w,v) and (v,q). Explain why T_1 is a spanning tree. Compute the tree solution $x = x_{T_1}$ associated with T_1.

Solution: T_1 contains all nodes in D, it is connected and contains no cycle; so it is a spanning tree. To compute $x = x_{T_1}$, use leaf elimination, and compute (for instance in the order (v,q), (w,v), (w,p) and (u,w)). This gives $x_{vq} = 3$, $x_{wv} = 3$, $x_{wp} = 1$, $x_{uw} = 1$ and $x_{uv} = x_{vp} = x_{pq} = 0$.

□

3b

Use the network simplex algorithm to find an optimal solution, and the optimal value, of the network flow problem. Show the computations.

Solution: Start with the spanning tree T_1 and x above. Use node u as the root. We compute y using $y_j = y_i + c_{ij}$ for each edge (i,j) in T_1: $y_u = 0$, $y_w = 1$, $y_v = 3$, $y_p = 7$, $y_q = 8$. Next we compute z from $z_{ij} = y_i + c_{ij} - y_j$ for each edge (i,j) (this is zero for edges in the tree). We get: $z_{uv} = 1$, $z_{vp} = -1$, $z_{pq} = 5$. So not optimal.

Pivot: Take (v,p) into the basis. Let $x_{vp} = \epsilon$, so $x_{wv} = 3+\epsilon$, $x_{wp} = 1-\epsilon$. Maximum ϵ is 1. New spanning tree is T_2 with edges (u,w), (w,v), (v,p) and (v,q). The tree solution is now $x_{uw} = 1$, $x_{wv} = 4$, $x_{vp} = 1$ and $x_{vq} = 3$ and the other flows are 0. Dual solution: $y_u = 0$, $y_w = 1$, $y_v = 3$, $y_p = 6$, $y_q = 8$, and $z_{uv} = 1$, $z_{wp} = 1$, $z_{pq} = 4$. Since z is nonnegative, the present tree solution x is optimal. The optimal value is 27.

□

3c

Let A be the incidence matrix of the directed graph D above (i.e., the coefficient matrix of the flow balance equations). What is the rank of A?

(Continued on page 5.)

Consider again the solution $x = x_{T_1}$ from question a). Assume that there are (feasible) flow vectors x^1 and x^2 in the network flow problem such that

$$x = 0.2\,x^1 + 0.8\,x^2$$

What can you say about x^1 and x^2? Explain.

Solution: From the network flow theory we know that A has rank $n-1 = 5-1 = 4$ and that every (feasible) tree solution is a basic feasible solution in the network flow LP problem. Moreover, from convexity, we know that a point is a basic feasible solution if and only if it is an extreme point of the feasible set (a polyhedron). Therefore x above cannot be written as a convex combination of two other feasible points, so the only possibility is that $x^1 = x^2 = x$. This can also be shown directly using the constraints in the flow problem and a small convexity argument. □

Problem 4

Let α and β be real numbers and consider the matrix game given by the following 3×4 matrix A (which depends on α, β)

$$A = \begin{bmatrix} 3 & 8 & 7 & 11 \\ 2 & 4 & \alpha & \beta \\ 3 & 1 & 6 & 7 \end{bmatrix}.$$

4a

Determine the set H consisting of all $(\alpha, \beta) \in \mathbb{R}^2$ such that (row) 2 is a pure minmax strategy for the row player R and (column) 3 is a pure maxmin strategy for the column player K.

Solution: From the lecture on game theory we know that: (row) 2 is a pure minmax strategy for player R and (column) 3 is a pure maxmin strategy for player K if and only if the pair $(2,3)$ is a saddle point of A. But this means that $a_{23} = \alpha$ is the minimum in column 3 and the maximum in row 2, so

$$\alpha \le 6,\ \alpha \ge 4,\ \alpha \ge \beta$$

or equivalently that

$$4 \le \alpha \le 6,\ \alpha \ge \beta.$$

□

Problem 5

Let H be a real $m \times n$ matrix and let $c \in \mathbb{R}^m$. Assume that a vector $\bar{z} \in \mathbb{R}^n$ satisfies $H\bar{z} = c$ and $\bar{z} \ge O$ (so $\bar{z}$ is nonnegative).

(Continued on page 6.)

5a

Prove that $Hz = c$ has a nonnegative solution z with at most m positive variables. (Hint: consider the LP problem min $\{\sum_{i=1}^{m} w_i : Hz + w = c,\ z \geq O,\ w \geq O\}$.)

Solution: In the mentioned LP there is a feasible solution $z = \bar{z}$, $w = O$, and therefore the optimal value must be 0 (and $z = \bar{z}$, $w = O$ is optimal). By the fundamental theorem of LP there is an optimal basic solution (z, w): in this solution $w = O$ and there are m basic variables (as the coefficient matrix $[H \;\; I]$ has rank m) and therefore at most m of the components of z are positive in this solution. Moreover, $b = Hz + w = Hz$ and the result holds. □

Consider a linear system $Ax \leq b$, where A is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

5b

State Farkas' lemma for the system $Ax \leq b$. Show that if $Ax \leq b$ is inconsistent (meaning: has no solution), then $Ax \leq b$ contains a subsystem with at most $n + 1$ inequalities which is also inconsistent.

Solution: Farkas' lemma: $Ax \leq b$ has a solution if and only if $y^T b \geq 0$ for every $y \in \mathbb{R}^m$ with $y^T A = O$ and $y \geq O$.

Second part: Assume $Ax \leq b$ is inconsistent. By Farkas' lemma there is a $y \in \mathbb{R}^m$ with $y^T A = O$, $y \geq O$and $y^T b < 0$. We may assume that $y^T b = -1$ by suitable (nonnegative) scaling of y. But then y is a nonnegative vector satisfying the $n + 1$ linear equations $y^T A = O$ and $y^T b = -1$. By question a) above, this system of equations must have a nonnegative solution y^* with at most $n+1$ positive components. But then the subsystem of $Ax \leq b$ consisting of the inequalities corresponding to the $n + 1$ positive components of y^* must be inconsistent; again due to Farkas' lemma.

□