# UNIVERSITY OF OSLO

## Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3100 — Linear Optimization

Day of examination: May 28, 2014

Examination hours: 09.00 – 13.00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are 10 questions each with roughly the same weight.

*Solutions.*

## Problem 1

Assume you have solved phase I of an LP problem (using the standard approach with $x_0$ as the extra variable) and the optimal dictionary of phase I is:

$$\begin{array}{rcrcrcrcr} \xi & = & 0 & - & x_0 & & & & \\ \hline x_1 & = & 1 & & & + & 2x_4 & - & x_5 \\ x_2 & = & 2 & - & x_0 & & & - & x_5 \\ x_3 & = & 3 & & & - & x_4 & + & x_5 \end{array}$$

As usual all variables are nonnegative. The original objective function to be maximized is $f(x) = x_1 - x_2$.

### 1a

Use the information above to solve the LP problem (i.e., phase II) using the simplex algorithm and find an optimal solution, and the optimal value.

*Solution: We see from phase I that the original problem is feasible (as $x_0 = 0$), and we compute the objective function: $\eta = f(x) = x_1 - x_2 = -1 + 2x_4$. This gives the dictionary*

$$\begin{array}{rcrcrcr} \eta & = & -1 & + & 2x_4 & & \\ \hline x_1 & = & 1 & + & 2x_4 & - & x_5 \\ x_2 & = & 2 & & & - & x_5 \\ x_3 & = & 3 & - & x_4 & + & x_5 \end{array}$$

(Continued on page 2.)

*Pivot: $x_4$ in and $x_3$ out:*

$$\begin{array}{rcrcrcr} \eta & = & 5 & - & 2x_3 & + & 2x_5 \\ \hline x_1 & = & 7 & - & 2x_3 & + & x_5 \\ x_2 & = & 2 & & & - & x_5 \\ x_4 & = & 3 & - & x_3 & + & x_5 \end{array}$$

*Pivot: $x_5$ in and $x_2$ out:*

$$\begin{array}{rcrcrcr} \eta & = & 9 & - & 2x_3 & - & 2x_2 \\ \hline x_1 & = & 9 & - & 2x_3 & - & x_2 \\ x_5 & = & 2 & & & - & x_2 \\ x_4 & = & 5 & - & x_3 & - & x_2 \end{array}$$

*So the optimal value is 9, and an optimal solution is $x_1 = 9$, $x_2 = 0$, $x_3 = 0$, $x_4 = 5$, $x_5 = 2$.*

□

Consider an LP problem (with nonnegative variables) where the simplex algorithm leads to the following dictionary:

$$\begin{array}{rcrcrcr} \eta & = & 1 & + & 2x_4 & & \\ \hline x_1 & = & 3 & + & x_4 & & \\ x_2 & = & 0 & - & x_4 & & \\ x_3 & = & 8 & & & + & 2x_5 \end{array}$$

### 1b

Find all optimal solutions of this problem. Which of these optimal solutions are degenerate?

*Solution: The present solution is degenerate, but still we can make a pivot: $x_4$ enters and $x_2$ leaves. This gives*

$$\begin{array}{rcrcrcr} \xi & = & 1 & - & 2x_2 & & \\ \hline x_1 & = & 3 & - & x_2 & & \\ x_4 & = & 0 & - & x_2 & & \\ x_3 & = & 8 & & & + & 2x_5 \end{array}$$

*Therefore all optimal solutions are $x_1 = 3$, $x_2 = 0$, $x_3 = 8 + 2x_5$, $x_4 = 0$, $x_5$ is free. All these solutions are degenerate as $x_2 = 0$ in each one.*

□

## Problem 2

Let $A$ be an $m \times n$ matrix, $b \in \mathbb{R}^m$ and $u \in \mathbb{R}^n_+$. $O$ denotes the zero vector. Consider the LP problem

$$\text{(P)} \quad \begin{array}{ll} \max & \sum_{j=1}^n x_j \\ \text{subject to} & \\ & Ax \leq b \\ & O \leq x \leq u. \end{array}$$

*(Continued on page 3.)*

### 2a

Find the dual of (P).

*Solution: Let* $e = (1, \dots, 1) \in \mathbb{R}^n$. *(As usual vectors are column vectors, and identified with* $n$*-tuples.) Problem (P) may be written*

$$\max\{e^T x : Ax \le b, Ix \le u,\ x \ge O\} \quad = \max\{e^T x : \begin{bmatrix} A \\ I \end{bmatrix} x \le \begin{bmatrix} b \\ u \end{bmatrix},\ x \ge O\}.$$

*The dual is*

$$\min\{b^T y_1 + u^T y_2 : A^T y_1 + I^T y_2 \ge e,\ y_1, y_2 \ge O\}$$

*i.e.*

$$\min\{b^T y_1 + u^T y_2 : A^T y_1 + y_2 \ge e,\ y_1, y_2 \ge O\}$$

□

Assume (for the remaining two questions) that $Au \le 2b$.

### 2b

Explain why (P) has an optimal solution.

*Solution: Then* $A((1/2)u) \le b$ *and* $O \le (1/2)u \le u$*, so (P) has a feasible solution. Moreover, (P) is not unbounded as the feasible set is bounded (or because the dual is feasible). By the fundamental theorem of LP it follows that (P) has an optimal solution.* □

Assume that we have given a nonnegative vector $z \in \mathbb{R}^m$ satisfying $z^T A \ge O$.

### 2c

Prove that $b^T z + \sum_{i=1}^m u_i$ is an upper bound on the optimal value of (P).

*Solution: Note that* $y_1 = z$, $y_2 = e$ *(where* $e$ *is the all ones vector) is a feasible solution of the dual of (P), as both vectors are nonnegative and* $A^T z + e \ge O + e = e$*. Therefore, by weak duality, the dual objective function value* $b^T z + e^T u = b^T z + \sum_{i=1}^m u_i$ *is an upper bound on the optimal value of (P).* □

## Problem 3

Consider the minimum cost network flow problem in the directed graph $D$ indicated in Figure 1. There are five nodes (vertices), $u$, $v$ etc. The numbers along the edges are the costs (per unit flow). The supply/demand is given by $b_w = 2$, $b_q = -2$ and $b_u = b_v = b_p = 0$. Let the spanning tree $T$ consist of all nodes and the edges $(w, u)$, $(w, v)$, $(v, p)$ and $(p, q)$.

Compute the tree solution $x = x^T$ associated with the tree $T$, and use the network flow simplex algorithm to prove that $x$ is an optimal solution of the minimum cost network flow problem. What is the optimal value?

*(Continued on page 4.)*

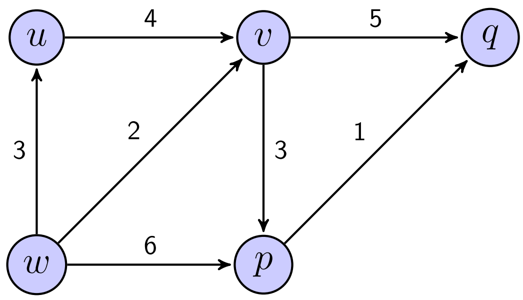


Figure 1: The directed graph $D$.

*Solution: Using the tree solution algorithm (leaf elimination) we obtain:* $x_{wv} = x_{vp} = x_{pq} = 2$ *and the remaining four variables are all 0.*

*We compute the dual solution corresponding to* $T$*: Use node* $w$ *as the root. We compute* $y$ *using* $y_j = y_i + c_{ij}$ *for each edge* $(i, j)$ *in* $T$*:* $y_w = 0$, $y_v = 2$, $y_p = 5$, $y_q = 6$, $y_u = 3$. *Next we compute* $z$ *from* $z_{ij} = y_i + c_{ij} - y_j$ *for each edge* $(i, j)$ *(this is zero for edges in the tree). We get:* $z_{uv} = 5$, $z_{wp} = 1$, $z_{vq} = 1$. *Since* $z \geq O$*, the present solution is optimal, so* $x$ *is optimal solution of the minimum cost network flow problem. The optimal value is 12.*

□

# Problem 4

## 4a

Give the definitions of (i) the *convex hull*, (ii) *polytope*, and (iii) *polyhedron*. State (without proof) an important result from convexity which relates polytopes and polyhedra.

*Solution: See lecture notes on convexity.*

Consider the linear system

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & \leq & 2 \\ x_1 & - & x_2 & + & x_3 & \leq & 2 \\ -x_1 & + & 3x_2 & - & x_3 & \leq & 2 \\ & & x_2 & + & x_3 & \leq & 2 \\ & & & & x_3 & \leq & 1 \end{array} \tag{1}$$

## 4b

Eliminate $x_1$ using Fourier-Motzkin elimination and find the resulting system (in $x_2, x_3$) and the bounds on $x_1$ expressed in terms of $x_2$ and $x_3$. Finally, determine the set

$$K = \{x_3 \in \mathbb{R} : (x_1, x_2, x_3) \text{ satisfies (1) for some } x_1, x_2 \in \mathbb{R}\}$$

*(Continued on page 5.)*

*Solution: The inequalities in (1) involving* $x_1$ *are*

$$\begin{array}{rcl} x_1 & \le & 2 - x_2 - x_3 \\ x_1 & \le & 2 + x_2 - x_3 \\ 3x_2 - x_3 - 2 & \le & x_1 \end{array}$$

*so we get*

$$3x_2 - x_3 - 2 \le x_1 \le \min\{2 - x_2 - x_3, 2 + x_2 - x_3\}$$

*and the new system in* $x_2, x_3$ *is*

$$\begin{array}{rcrcl} 4x_2 & & & \le & 4 \\ 2x_2 & & & \le & 4 \\ x_2 & + & x_3 & \le & 2 \\ & & x_3 & \le & 1 \end{array}$$

*or equivalently*

$$\begin{array}{rcrcl} x_2 & & & \le & 1 \\ x_2 & + & x_3 & \le & 2 \\ & & x_3 & \le & 1 \end{array}$$

*Here* $x_2 + x_3 \le 2$ *is redundant, so the answer is the system:*

$$\begin{array}{rcl} x_2 & & \le 1 \\ & x_3 & \le 1 \end{array}$$

*In order to find the set* $K$ *we proceed and eliminate* $x_2$*. We get*

$$x_2 \le 1$$

*and the new system* $x_3 \le 1$*. By Fourier-Motzkin theory this proves that*

$$K = \{x_3 : x_3 \le 1\}.$$

□

## Problem 5

Let

$$A = \begin{bmatrix} M & O \\ N & A_1 \end{bmatrix}$$

where (i) $M$ is an invertible $k \times k$ matrix, (ii) $O$ is the zero matrix, (iii) $N$ is a $p \times k$ matrix, and (iv) $A_1$ is a $p \times n$ matrix with rank $p$, so $A_1$ has full row rank. (Recall that a *basis* in an $r \times s$ matrix $W$ of rank $r$ is an invertible $r \times r$ submatrix of $W$.)

*(Continued on page 6.)*

### 5a

Show that $B$ is a basis in $A$ if and only if $B$ has the form

$$B = \begin{bmatrix} M & O \\ N & B_1 \end{bmatrix}$$

where $B_1$ is a basis in $A_1$.

*Solution: $A$ is of size $(k+p) \times (k+n)$. A basis $B$ in $A$ is obtained by selecting $k+p$ columns from $A$ such that the corresponding submatrix $B$ is invertible. We claim that one must choose all the first $k$ columns in $A$, otherwise $B$ will not be invertible. In fact, we we choose $k' < k$ among the first $k$ columns (of $A$), then the corresponding submatrix $M'$ of $M$ has only $k'$ columns that are nonzero. Thus the column rank of $M'$ is at most $k'$, and since row rank and column rank are the same, the row rank of $M'$ is $k'$. The row rank of the submatrix of $A$ formed by the $p$ last rows is at most $p$. Thus, the rows of $B$ span a subspace of dimension at most $p + k' < p + k$. This shows that we have to choose all the first $k$ columns of $A$ to get a basis. The remaining $p$ columns are chosen from the remaining columns of $A$, and this gives the structure above where $B_1$ is $p \times p$. Finally, if $B$ is a basis, then $B_1$ must be invertible as $\det B = \det A_1 \cdot \det B_1$ and both $\det A_1$ and $\det B$ are nonzero. Conversely, if $B_1$ is a basis in $A_1$, the same determinant formula, shows that $B$ is a basis, as desired.*

Consider the polyhedron

$$P = \{x \in \mathbb{R}^{k+n} : Ax = b,\ x \geq O\}$$

where the variable vector $x$ and $b$ may be partitioned as

$$x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix},\ b = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}.$$

with $x^1 \in \mathbb{R}^k$, $x^2 \in \mathbb{R}^n$, $b^1 \in \mathbb{R}^k$ and $b^2 \in \mathbb{R}^p$.

### 5b

Find a description of all extreme points of $P$ using the information about $A$.

*Solution: From the notes on convexity we know that $x$ is an extreme point of $P$ if and only if $x$ is a basic feasible solutions in the system $Ax = b$. Thus we need to find all bases in $A$, and find the corresponding basic solution. For this we use the property in the previous question and the structure of a basis $B$. Using the partitioning of $x$ above, let $x_1^2$ be the subvector of $x^2$ containing the components corresponding to the columns selected to form $B_1$. Then the equation system for finding the basic variables is*

$$\begin{bmatrix} M & O \\ N & B_1 \end{bmatrix} \begin{bmatrix} x^1 \\ x_1^2 \end{bmatrix} = \begin{bmatrix} b^1 \\ b^2 \end{bmatrix}$$

*which gives $x^1 = M^{-1}b^1$ and therefore $x_1^2 = B_1^{-1}(b^2 - NM^{-1}b^1)$. If this solution is nonnegative it gives a basic feasible solution, and therefore an extreme point. Conversely, any extreme point has this form.* □