UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3100 — Linear Optimization

Day of examination: Monday, June 6th, 2016

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

1a

Consider the LP problem

$$
\begin{aligned}
&\text{maximize } -x_1 + 3x_2 + 2x_3\\
&\text{subject to}\\
&-x_1 + x_2 + 2x_3 \le 2,\\
&-3x_1 + 2x_2 + x_3 \le 1,\\
&8x_1 - 3x_2 + 2x_3 \le 2,\\
&x_1, x_2, x_3 \ge 0.
\end{aligned}
\tag{1}
$$

Use the simplex algorithm to find the optimal solution.

1b

Determine the dual problem of (1). Moreover, find an optimal solution of the dual problem.

1c

Consider the primal problem

$$
\begin{aligned}
&\text{maximize } 3x_1 + 2x_2 + x_3\\
&\text{subject to}\\
&x_1 - x_2 + x_3 \le 4,\\
&2x_1 + x_2 + 3x_3 \le 6,\\
&-x_1 + 2x_3 \le 3,\\
&x_1 + x_2 + x_3 \le 8,\\
&x_1, x_2, x_3 \ge 0,
\end{aligned}
\tag{2}
$$

(Continued on page 2.)

and the corresponding dual problem

$$\begin{aligned}
&\text{minimize } 4y_1 + 6y_2 + 3y_3 + 8y_4\\
&\text{subject to}\\
&y_1 + 2y_2 - y_3 + y_4 \geq 3,\\
&-y_1 + y_2 + y_4 \geq 2,\\
&y_1 + 3y_2 + 2y_3 + y_4 \geq 1,\\
&y_1, y_2, y_3 \geq 0.
\end{aligned} \tag{3}$$

State the complementary slackness conditions for optimality of a feasible solution $x \in \mathbb{R}^3$ of the primal problem (2) and a feasible solution $y \in \mathbb{R}^4$ of the dual problem (3).

1d

Suppose $(x_1, x_2, x_3) = (0, 6, 0)$ is optimal for the primal problem (2). Use the complementary slackness conditions to solve the dual problem.

Problem 2

A company produces food products A and B using machines M_1 and M_2. One ton of product A requires 1 hour of processing on machine M_1 and 2 hours on machine M_2. One ton of product B requires 3 hours of processing on M_1 and 1 hour on M_2. Each day machine M_1 has available 9 hours of processing time, while machine M_2 has available 8 hours. Each ton of product produced (of either type) yields \$1 million profit.

2a

The problem is to decide how much of each food product should the company make per day to maximize profit. Formulate this optimization problem as a linear programming problem. Graph the feasible region F.

2b

Define what it means for a set $C \subset \mathbb{R}^n$ $(n \geq 1)$ to be convex. Given a set $P \subset \mathbb{R}^n$, define the convex hull of P, $\text{conv}(P)$. What is a polytope?

2c

Identify four extreme points p_1, p_2, p_3, p_4 such that the feasible region F in **2a** can be written as $\text{conv}(\{p_1, p_2, p_3, p_4\})$. A known theorem states that $x \in F$ is a basic solution (in the LP sense) if and only if x is an extreme point of F. Use this to determine the optimal (basic) solution to the linear programming problem formulated in **2a**.

(Continued on page 3.)

Problem 3

3a

Consider a general game defined by a matrix $A = \{a_{i,j}\}_{i,j} \in \mathbb{R}^{m \times n}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. What do we mean by (pure) minmax and maxmin strategies and the game's value?

Determine the minmax and maxmin strategies and value for the game given by

$$A = \begin{pmatrix} 2 & 8 & 6 & 11 \\ 2 & 3 & 4 & 2 \\ 1 & 1 & 5 & 4 \end{pmatrix} \in \mathbb{R}^{3 \times 4}. \tag{4}$$

3b

Consider a game given by a matrix $A = \{a_{i,j}\} \in \mathbb{R}^{m \times n}$. Explain (define) what we mean by a saddle point. Using the definition of a saddle point, verify that the strategies found in **3a** for (4) constitute a saddle point.

3c

Given a general matrix game defined by $A = \{a_{i,j}\} \in \mathbb{R}^{m \times n}$, suppose the row player R has a pure minmax strategy r, the column player K has a pure maxmin strategy s, and that the game has a value V. Show that (r, s) is a saddle point and that the value of the game is $V = a_{r,s}$.

THE END