UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT-INF3100 — Linear Optimization

Day of examination: Monday, June 6th, 2016

Examination hours: 14.30 – 18.30

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

1a

Consider the LP problem

$$\begin{aligned}
&\text{maximize } -x_1 + 3x_2 + 2x_3 \\
&\text{subject to} \\
&-x_1 + x_2 + 2x_3 \leq 2, \\
&-3x_1 + 2x_2 + x_3 \leq 1, \\
&8x_1 - 3x_2 + 2x_3 \leq 2, \\
&x_1, x_2, x_3 \geq 0.
\end{aligned} \tag{1}$$

Use the simplex algorithm to find the optimal solution.

<u>Answer:</u> Initial dictionary

$$\begin{aligned}
\max \eta &= -x_1 + 3x_2 + 2x_3, \\
w_1 &= 2 + x_1 - x_2 - 2x_3, \\
w_2 &= 1 + 3x_1 - 2x_2 - x_3, \\
w_3 &= 2 - 8x_1 + 3x_2 - 2x_3.
\end{aligned}$$

We perform a pivot step with x_2 into the basis and w_2 out of the basis (so $x_2 = \frac{1}{2} + \frac{3}{2}x_1 - \frac{1}{2}x_3 - \frac{1}{2}w_2$), resulting in the dictionary

$$\begin{aligned}
\max \eta &= \tfrac{3}{2} + \tfrac{7}{2}x_1 + \tfrac{1}{2}x_3 - \tfrac{3}{2}w_2, \\
w_1 &= \tfrac{3}{2} - \tfrac{1}{2}x_1 - \tfrac{3}{2}x_3 + \tfrac{1}{2}w_2, \\
x_2 &= \tfrac{1}{2} + \tfrac{3}{2}x_1 - \tfrac{1}{2}x_3 - \tfrac{1}{2}w_2, \\
w_3 &= \tfrac{7}{2} - \tfrac{7}{2}x_1 - \tfrac{7}{2}x_3 - \tfrac{3}{2}w_2.
\end{aligned}$$

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We perform a pivot step with x_1 into the basis and w_3 out of the basis (so $x_1 = 1 - x_3 - \frac{3}{7}w_2 - \frac{2}{7}w_3$), resulting in the dictionary

$$\begin{aligned} \max \eta &= 5 - 3x_3 - 3w_2 - w_3, \\ w_1 &= 1 - x_3 + \tfrac{10}{14}w_2 + \tfrac{1}{7}w_3, \\ x_2 &= 2 - 2x_3 - \tfrac{8}{7}w_2 - \tfrac{3}{7}w_3, \\ x_1 &= 1 - x_3 - \tfrac{3}{7}w_2 - \tfrac{2}{7}w_3. \end{aligned} \tag{2}$$

This is an optimal dictionary. The optimal solution is

$$(x_1, x_2, x_3) = (1, 2, 0),$$

with corresponding optimal value 5.

1b

Determine the dual problem of (1). Moreover, find an optimal solution of the dual problem.

Answer: The dual problem reads

$$\begin{aligned} &\text{minimize } 2y_1 + y_2 + 2y_3 \\ &\text{subject to} \\ &-y_1 - 3y_2 + 8y_3 \geq -1, \\ &y_1 + 2y_2 - 3y_3 \geq 3, \\ &2y_1 + y_2 + 2y_3 \geq 2, \\ &y_1, y_2, y_3 \geq 0. \end{aligned} \tag{3}$$

Using the "negative transpose property" (complementarity between x_j and z_j and between y_i and w_i), we read off the dual solution from the optimal dictionary (2) for the primal problem. The optimal dual solution is

$$(y_1, y_2, y_3) = (0, 3, 1),$$

with dual objective value is 5.

1c

Consider the primal problem

$$\begin{aligned} &\text{maximize } 3x_1 + 2x_2 + x_3 \\ &\text{subject to} \\ &x_1 - x_2 + x_3 \leq 4, \\ &2x_1 + x_2 + 3x_3 \leq 6, \\ &-x_1 + 2x_3 \leq 3, \\ &x_1 + x_2 + x_3 \leq 8, \\ &x_1, x_2, x_3 \geq 0, \end{aligned} \tag{4}$$

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and the corresponding dual problem

$$\begin{aligned}
&\text{minimize}\, 4y_1 + 6y_2 + 3y_3 + 8y_4\\
&\text{subject to}\\
&y_1 + 2y_2 - y_3 + y_4 \geq 3,\\
&-y_1 + y_2 + y_4 \geq 2,\\
&y_1 + 3y_2 + 2y_3 + y_4 \geq 1,\\
&y_1, y_2, y_3 \geq 0.
\end{aligned} \tag{5}$$

State the complementary slackness conditions for optimality of a feasible solution $x \in \mathbb{R}^3$ of the primal problem (4) and a feasible solution $y \in \mathbb{R}^4$ of the dual problem (5).

Answer: Denote by (w_1, w_2, w_3, w_4) the primal slack variables,

$$\begin{aligned}
w_1 &= 4 - x_1 + x_2 - x_3,\\
w_2 &= 6 - 2x_1 - x_2 - 3x_3,\\
w_3 &= 3 + x_1 - 2x_3,\\
w_4 &= 8 - x_1 - x_2 - x_3,
\end{aligned}$$

and by (z_1, z_2, z_3) the dual slack variables,

$$\begin{aligned}
z_1 &= -3 + y_1 + 2y_2 - y_3 + y_4,\\
z_2 &= -2 - y_1 + y_2 + y_4,\\
z_3 &= -1 + y_1 + 3y_2 + 2y_3 + y_4.
\end{aligned}$$

Then x is optimal for the primal problem and y is optimal for the dual problem if and only if

$$x_j z_j = 0, \qquad j = 1, 2, 3, \tag{6}$$

and

$$w_i y_i = 0, \quad i = 1, 2, 3, 4. \tag{7}$$

1d

Suppose $(x_1, x_2, x_3) = (0, 6, 0)$ is optimal for the primal problem (4). Use the complementary slackness conditions to solve the dual problem.

Answer: If $(x_1, x_2, x_3) = (0, 6, 0)$ is optimal for the primal problem (4), we compute

$$w_1 = 10, \quad w_2 = 0, \quad w_3 = 3, \quad w_4 = 2,$$

and the strict inequalities $w_1 > 0$, $w_3 > 0$, $w_4 > 0$ imply via (7) that $y_1 = y_3 = y_4 = 0$. Moreover, $x_2 > 0$ implies via (6) that $z_2 = 0$, and thus (since $y_1 = y_3 = y_4 = 0$) it follows that $y_2 = 2$. One can easily check that $y = (0, 2, 0, 0)$ is feasible, and we thus conclude that y is optimal.

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Problem 2

A company produces food products A and B using machines M_1 and M_2. One ton of product A requires 1 hour of processing on machine M_1 and 2 hours on machine M_2. One ton of product B requires 3 hours of processing on M_1 and 1 hour on M_2. Each day machine M_1 has available 9 hours of processing time, while machine M_2 has available 8 hours. Each ton of product produced (of either type) yields \$1 million profit.

2a

The problem is to decide how much of each food product should the company make per day to maximize profit. Formulate this optimization problem as a linear programming problem. Graph the feasible region F.

Answer: Denote by x_1 the number of tons produced of product A, and by x_2 the number of tons produced of product B. Then the LP formulation reads

$$\begin{aligned}&\text{maximize}\, x_1 + x_2\\ &\text{subject to}\\ &x_1 + 3x_2 \leq 9,\\ &2x_1 + x_2 \leq 8,\\ &x_1, x_2 \geq 0.\end{aligned} \tag{8}$$

The feasible region is plotted in Figure 1.

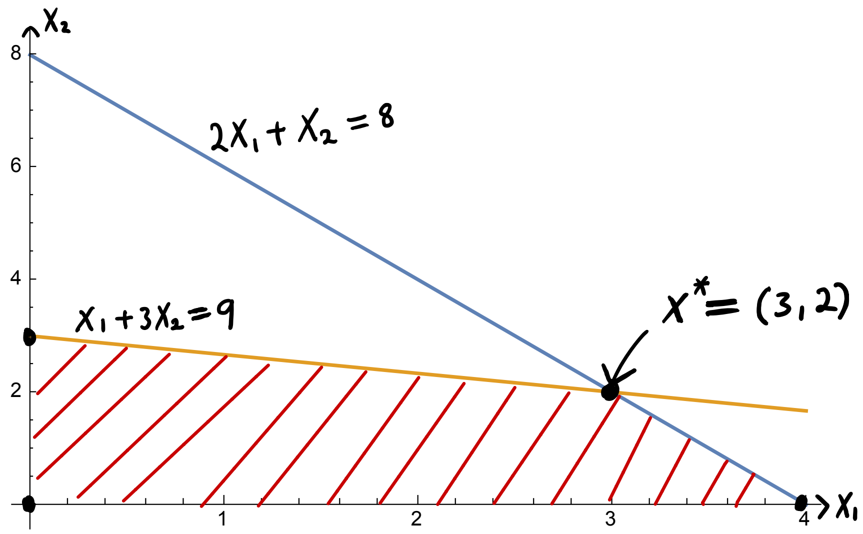

Figure 1: Plot of the feasible region in Problem 2a)

2b

Define what it means for a set $C \subset \mathbb{R}^n$ ($n \geq 1$) to be convex. Given a set $P \subset \mathbb{R}^n$, define the convex hull of P, conv(P). What is a polytope?

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Answer: The set C is convex if for all $x_1, x_2 \in C$,

$$(1-\lambda)x_1 + \lambda x_2 \in C, \quad \forall \lambda \in [0,1].$$

The set conv(P) is the intersection of all convex sets containing P, i.e., the smallest convex set containing P. A polytype is a set that is the convex hull of a finite number of points in $\mathbb{R}^n$.

2c

Identify four extreme points p_1, p_2, p_3, p_4 such that the feasible region F in **2a** can be written as $\operatorname{conv}(\{p_1, p_2, p_3, p_4\})$. A known theorem states that $x \in F$ is a basic solution (in the LP sense) if and only if x is an extreme point of F. Use this to determine the optimal (basic) solution to the linear programming problem formulated in **2a**.

Answer: Take $p_1 = (0,0)$, $p_2 = (4,0)$, $p_3 = (3,2)$, and $p_4 = (0,3)$. Then $F = \operatorname{conv}(\{p_1, p_2, p_3, p_4\})$. Clearly, p_1, p_2, p_3, p_4 are extreme points (i.e., they cannot be written as convex combinations of other points in F), and evaluating the objective function $x_1 + x_2$ at these points we obtain

$$0+0=0, \quad 4+0=4, \quad 3+2=5, \quad 0+3=3.$$

Hence the optimal solution is $x^\star = (3,2)$, with objective value 5.

Problem 3

3a

Consider a general game defined by a matrix $A = \{a_{i,j}\}_{i,j} \in \mathbb{R}^{m\times n}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. What do we mean by (pure) minmax and maxmin strategies and the game's value?

Determine the minmax and maxmin strategies and value for the game given by

$$A = \begin{pmatrix} 2 & 8 & 6 & 11 \\ 2 & 3 & 4 & 2 \\ 1 & 1 & 5 & 4 \end{pmatrix} \in \mathbb{R}^{3\times 4}. \tag{9}$$

Answer: We call $1 \le r \le m$ a minmax strategy (for the row player R) if

$$\min_{i=1,\ldots m} \max_{j=1,\ldots,n} a_{i,j} = \max_{j=1,\ldots,n} a_{r,j}.$$

We call $1 \le s \le n$ a maxmin strategy (for the column player K) if

$$\max_{j=1,\ldots,n} \min_{i=1,\ldots m} a_{i,j} = \min_{i=1,\ldots,m} a_{i,s}.$$

The game has a value V if

$$\min_{i=1,\ldots m} \max_{j=1,\ldots,n} a_{i,j} = \max_{j=1,\ldots,n} \min_{i=1,\ldots m} a_{i,j} = V.$$

For (9) we compute $\max_j a_{1,j} = 11$, $\max_j a_{2,j} = 4$, $\max_j a_{3,j} = 5$, and hence $r = 2$ is a minmax strategy.

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Similarly, $\min_i a_{i,1} = 1$, $\min_i a_{i,2} = 1$, $\min_i a_{i,3} = 4$, $\min_i a_{i,4} = 2$, and hence $s = 3$ is a maxmin strategy.

The game's value is 4 since

$$\min_i \max_j a_{i,j} = \max_j \min_i a_{i,j} = 4.$$

3b

Consider a game given by a matrix $A = \{a_{i,j}\} \in \mathbb{R}^{m\times n}$. Explain (define) what we mean by a saddle point. Using the definition of a saddle point, verify that the strategies found in **3a** for (9) constitute a saddle point.

Answer: A saddle point is a pair (r, s) of strategies (for R and K) satisfying

$$a_{r,j} \le a_{r,s} \le a_{i,s}, \qquad i = 1, \dots, m,\ j = 1, \dots, n.$$

so $a_{r,s}$ is smallest in its column (s) and largest in its row (r). For (9) it is easily verified that $a_{2,3} = 4$ satisfies

$$a_{2,j} \le 4 \le a_{i,3}, \qquad i = 1, \dots, 3,\ j = 1, \dots, 4.$$

3c

Given a general matrix game defined by $A = \{a_{i,j}\} \in \mathbb{R}^{m\times n}$, suppose the row player R has a pure minmax strategy r, the column player K has a pure maxmin strategy s, and that the game has a value V. Show that (r, s) is a saddle point and that the value of the game is $V = a_{r,s}$.

Answer: For any $i = 1, \dots, m$,

$$a_{i,s} \ge \min_{i=1,\dots m} a_{i,s} = \max_{j=1,\dots,n} \min_{i=1,\dots m} a_{i,j},$$

where the equality comes from s being a maxmin strategy for K. Since the game has a value V,

$$\max_{j=1,\dots,n} \min_{i=1,\dots m} a_{i,j} = V = \min_{i=1,\dots m} \max_{j=1,\dots,n} a_{i,j}.$$

Moreover, since r is a minmax strategy for R,

$$\min_{i=1,\dots m} \max_{j=1,\dots,n} a_{i,j} = \max_{j=1,\dots,n} a_{r,j} \ge a_{r,j},$$

where the latter inequality holds for any $j = 1, \dots, n$.

Summarizing, we have shown that

$$a_{r,j} \le V \le a_{i,s}, \qquad i = 1, \dots, m,\ j = 1, \dots n,$$

and taking $(i, j) = (r, s)$ we arrive at $V = a_{r,s}$. This concludes the proof.

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