UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT-INF3100 — Linear optimization

Day of examination: 01 June 2017

Examination hours: 0900 – 1300

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1

Consider the LP problem (P)

$$\begin{array}{rrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & -x_1 + 3x_2 & \leq 3, \\ & 2x_1 + x_2 & \leq 8, \\ & x_1, x_2 & \geq 0. \end{array}$$

1a

Use the simplex method to find an optimal solution and the corresponding objective value.

1b

Make a plot of the feasible region for (P) and indicate your optimal solution.

1c

(i) Let (P′) be the LP problem formed by adding the constraint, $x_1+3x_2 \leq 10$, to (P). What is the optimal objective value of (P′)?
(ii) Let (P″) be the LP problem formed by replacing the objective function of (P) by $x_1 + 3x_2$. What is the optimal objective value of (P″)?

1d

What is the dual problem (D) of (P)?

(Continued on page 2.)

1e

What is an optimal solution to (D) and what is the corresponding optimal value?

Problem 2

A matrix game is determined by a matrix $A = [a_{ij}]_{i=1,\ldots,m,j=1,\ldots,n}$. The row player (R) pays a_{ij} kroner to the column player (K) if R chooses option i and K chooses option j. R is playing with a randomized strategy $y = (y_1, \ldots, y_m)^T$, choosing option i with probability y_i, where $y_i \geq 0$ and $\sum_{i=1}^m y_i = 1$. Similarly, K is playing with a randomized strategy $x = (x_1, \ldots, x_n)^T$, where $x_j \geq 0$ and $\sum_{j=1}^n x_j = 1$.

2a

(i) If K uses a fixed strategy x, what is R's corresponding best defence, i.e., best corresponding strategy y?
(ii) If R adopts the strategy in (i) in defence of the strategy x chosen by K, what is K's best strategy x^*?

2b

Explain how part (ii) of the last problem can be formulated as the LP problem

$$\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & v \leq e_i^T A x, \quad i = 1, 2, \ldots, m, \\ & \displaystyle\sum_{j=1}^n x_j = 1, \\ & x_j \geq 0, \qquad j = 1, 2, \ldots, n, \end{array}$$

where $e_i \in \mathbb{R}^m$ is the vector of all zeros with 1 in the i-th position.

Problem 3

Consider the minimum cost network flow problem based on the directed graph shown in the figure. The number associated with each directed edge (i, j) is its cost $c_{i,j}$ (per unit flow). At each node i let b_i be its supply. The supplies are

$$b_u = 1, \quad b_v = -2, \quad b_w = -3, \quad b_p = 6, \quad b_q = -2.$$

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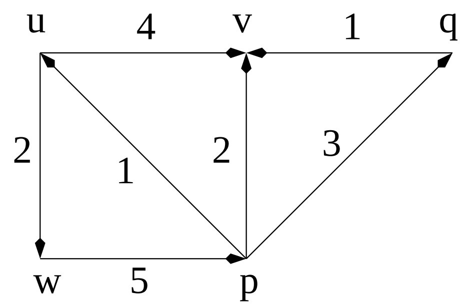

3a

Write down the flow balance equation at node i. Let T_1 be the spanning tree consisting of the edges

$$(u, w), \quad (p, u), \quad (p, q), \quad (q, v),$$

and all their nodes. Compute the tree solution x corresponding to T_1.

3b

Use the network simplex method to find an optimal solution and optimal value for the flow problem.

3c

In a general network flow problem, there may or may not be a unique optimal solution.
(i) Suppose the optimal solution is unique. If the supplies b_i are integers, will the flows x_{ij} in the optimal solution be integers?
(ii) Suppose there is more than one optimal solution. If the supplies b_i are integers, will the flows x_{ij} in an optimal solution be integers?
Explain your answers.

Good luck!