UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT-INF3100 — Linear optimization
Day of examination:	01 June 2017
Examination hours:	0900 – 1300

This problem set consists of 6 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

All 10 part questions will be weighted equally.

Problem 1

Consider the LP problem (P)

$$\begin{array}{rrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & -x_1 + 3x_2 & \leq 3, \\ & 2x_1 + x_2 & \leq 8, \\ & x_1, x_2 & \geq 0. \end{array}$$

1a

Use the simplex method to find an optimal solution and the corresponding objective value.

Answer: We introduce the slack variables $w_1 = 3 + x_1 - 3x_2$ and $w_2 = 8 - 2x_1 - x_2$, and then the initial dictionary is

$$\begin{array}{rcrcrcr} \eta & = & & & x_1 & + & 2x_2 \\ \hline w_1 & = & 3 & + & x_1 & - & 3x_2 \\ w_2 & = & 8 & - & 2x_1 & - & x_2 \end{array}$$

We can let x_2 go into the basis. When increasing x_2, we need $x_2 \leq 3/3$ to keep $w_1 \geq 0$, and $x_2 \leq 8/1$ to keep $w_2 \geq 0$, and so w_1 leaves the basis. Then, we express

$$x_2 = 1 + \frac{1}{3}x_1 - \frac{1}{3}w_1,$$

and with this substitution, the new dictionary is

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$$\begin{array}{rcrcrcr} \eta & = & 2 & + & (5/3)x_1 & - & (2/3)w_1 \\ \hline x_2 & = & 1 & + & (1/3)x_1 & - & (1/3)w_1 \\ w_2 & = & 7 & - & (7/3)x_1 & + & (1/3)w_1 \end{array}$$

Now x_1 goes into the basis, and w_2 leaves, and with the substitution,

$$x_1 = 3 - \frac{3}{7}w_2 + \frac{1}{7}w_1,$$

the new dictionary is

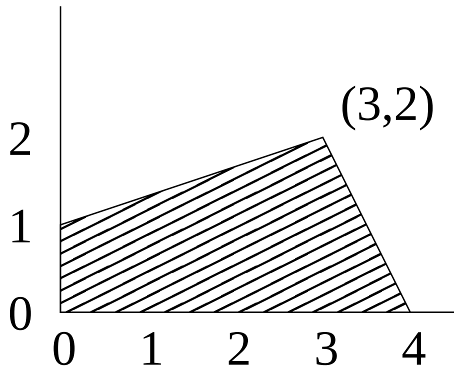
$$\begin{array}{rcrcrcr} \eta & = & 7 & - & (5/7)w_2 & - & (3/7)w_1 \\ \hline x_2 & = & 2 & - & (1/7)w_2 & - & (2/7)w_1 \\ x_1 & = & 3 & - & (3/7)w_2 & + & (1/7)w_1 \end{array}$$

This dictionary is optimal, and so an optimal solution is $w_1 = w_2 = 0$ and $x_1 = 3$, $x_2 = 2$, and the optimal objective value is 7.

1b

Make a plot of the feasible region for (P) and indicate your optimal solution.

Answer:



1c

(i) Let (P′) be the LP problem formed by adding the constraint, $x_1+3x_2 \leq 10$, to (P). What is the optimal objective value of (P′)?
(ii) Let (P″) be the LP problem formed by replacing the objective function of (P) by $x_1 + 3x_2$. What is the optimal objective value of (P″)?

Answer: (i) Since the set of points

$$\{(x_1, x_2) : x_1 + 3x_2 \leq 10\}$$

(Continued on page 3.)

contains the feasible region, the point (x_1, x_2) is also an optimal solution of (P′) and the optimal objective value of (P′) is the same as that of (P), i.e., 7.

(ii) The intersection of the line $x_1 + 3x_2 = 9$ with the feasible region is the single point $(3, 2)$. The objective function $x_1 + 3x_2$ increases in the direction perpendicular to this line, and so the previous optimal solution $(3, 2)$ is again optimal, but the new objective value is $1 \times 3 + 3 \times 2 = 9$.

1d

What is the dual problem (D) of (P)?

Answer:

$$\begin{array}{rrl} \text{minimize} & 3y_1 + 8y_2 & \\ \text{subject to} & -y_1 + 2y_2 & \geq 1, \\ & 3y_1 + y_2 & \geq 2, \\ & y_1, y_2 & \geq 0. \end{array}$$

1e

What is an optimal solution to (D) and what is the corresponding optimal value?

Answer: The dual variables (y_1, y_2) correspond to the slack variables (w_1, w_2) and the dual slack variables (z_1, z_2) correspond to the primal variables (x_1, x_2). Using the negative-transpose property of the simplex method, the dual dictionary corresponding to the optimal primal dictionary is

$$\begin{array}{rcrcrcr} -\zeta & = & 7 & - & 2z_2 & - & 3z_1 \\ \hline y_2 & = & 5/7 & + & (1/7)z_2 & + & (3/7)z_1 \\ y_1 & = & 3/7 & + & (2/7)z_2 & - & (1/7)z_1 \end{array}$$

This dictionary is also optimal, and the optimal dual solution is $z_1 = z_2 = 0$ and $y_1 = 3/7$ and $y_2 = 5/7$, and the optimal objective value is again 7.

Problem 2

A matrix game is determined by a matrix $A = [a_{ij}]_{i=1,\ldots,m, j=1,\ldots,n}$. The row player (R) pays a_{ij} kroner to the column player (K) if R chooses option i and K chooses option j. R is playing with a randomized strategy $y = (y_1, \ldots, y_m)^T$, choosing option i with probability y_i, where $y_i \geq 0$ and $\sum_{i=1}^m y_i = 1$. Similarly, K is playing with a randomized strategy $x = (x_1, \ldots, x_n)^T$, where $x_j \geq 0$ and $\sum_{j=1}^n x_j = 1$.

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2a

(i) If K uses a fixed strategy x, what is R's corresponding best defence, i.e., best corresponding strategy y?
(ii) If R adopts the strategy in (i) in defence of the strategy x chosen by K, what is K's best strategy x^*?

Answer: (i) R should find a y to minimize the expected payoff, i.e., find y, with $y_i \geq 0$ and $\sum_{i=1}^m y_i = 1$, that attains the minimum

$$\min_y \sum_i \sum_j y_i a_{ij} x_j = \min_y y^T A x.$$

(ii) K should find an x^* to maximize the expected payoff in (i), i.e. find x^*, that attains the maximium

$$\max_x \min_y y^T A x,$$

where both $x_j \geq 0$ and $\sum_{j=1}^n x_j = 1$, and $y_i \geq 0$ and $\sum_{i=1}^m y_i = 1$.

2b

Explain how part (ii) of the last problem can be formulated as the LP problem

$$\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & v \leq e_i^T A x, \quad i = 1, 2, \ldots, m, \\ & \displaystyle\sum_{j=1}^n x_j = 1, \\ & x_j \geq 0, \qquad j = 1, 2, \ldots, n, \end{array}$$

where $e_i \in \mathbb{R}^m$ is the vector of all zeros with 1 in the i-th position.

Answer: The set

$$\{y \in \mathbb{R}^m : y_i \geq 0, i = 1, \ldots, m, \sum_i y_i = 1\},$$

is a convex polyhedron in $\mathbb{R}^m$, a simplex with vertices e_i, $i = 1, \ldots, m$. Thus, $y^T A x$ attains its minimum at one of these vertices, and

$$\min_y y^T A x = \min_i e_i^T A x.$$

Thus it is sufficient to find x^* that attains the maximium

$$\max_x \min_i e_i^T A x.$$

We now let v be a lower bound on $\min_i e_i^T A x$ and formulate the problem as maximizing v, and this gives the stated LP problem.

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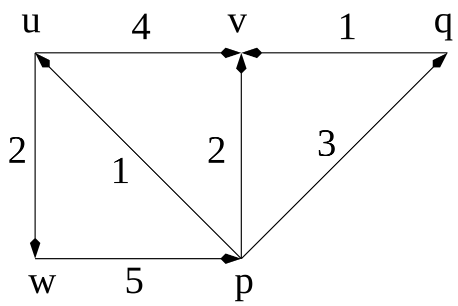

Problem 3

Consider the minimum cost network flow problem based on the directed graph shown in the figure. The number associated with each directed edge (i, j) is its cost $c_{i,j}$ (per unit flow). At each node i let b_i be its supply. The supplies are

$$b_u = 1, \quad b_v = -2, \quad b_w = -3, \quad b_p = 6, \quad b_q = -2.$$

3a

Write down the flow balance equation at node i. Let T_1 be the spanning tree consisting of the edges

$$(u, w), \quad (p, u), \quad (p, q), \quad (q, v),$$

and all their nodes. Compute the tree solution x corresponding to T_1.

Answer: The flow balance equation at node i is

$$\sum_{j:(i,j)\in E} x_{ij} - \sum_{k:(k,i)\in E} x_{ki} = b_i.$$

By 'tree solution' we mean that there is zero flow on edges not in T_1, i.e.,

$$x_{wp} = x_{uv} = x_{pv} = 0.$$

Using leaf elimination, for example, in the given order of the four edges of T_1, we use the supplies b_i to obtain

$$x_{uw} = 3, \quad x_{pu} = 2, \quad x_{pq} = 4, \quad x_{qv} = 2.$$

3b

Use the network simplex method to find an optimal solution and optimal value for the flow problem.

Answer: x above is a feasible solution. We compute the dual variables using $y_j = y_i + c_{ij}$ for each edge (i, j) in T_1. Use node u as the root and set $y_u = 0$. Then

$$y_w = 2, \quad y_p = -1, \quad y_q = 2, \quad y_v = 3.$$

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We now compute the dual slacks $z_{ij} = c_{ij} - (y_j - y_i)$ on the edges (i, j) not in T_1:

$$z_{wp} = 8, \quad z_{uv} = 1, \quad z_{pv} = -2.$$

Since z_{pv} is negative, x is not an optimal solution. So, we pivot. We take x_{pv} into the basis. If we increase x_{pv} from 0 to ϵ, then from the supplies, the new flows are as before, except

$$x_{pv} = \epsilon, \quad x_{pq} = 4 - \epsilon, \quad x_{qv} = 2 - \epsilon.$$

The maximum allowed increase in x_{pv} is therefore $\epsilon = 2$, and this makes $x_{qv} = 0$, and so x_{qv} leaves the basis. This gives us a new spanning tree T_2 with edges

$$(u, w), \quad (p, u), \quad (p, q), \quad (p, v),$$

and the new tree solution x is given by

$$x_{wp} = x_{uv} = x_{qv} = 0,$$

and

$$x_{uw} = 3, \quad x_{pu} = 2, \quad x_{pq} = 2, \quad x_{pv} = 2.$$

Now the dual variables, with $y_u = 0$, are

$$y_w = 2, \quad y_p = -1, \quad y_q = 2, \quad y_v = 1,$$

and then

$$z_{wp} = 8, \quad z_{uv} = 3, \quad z_{qv} = 2.$$

Now all the z_{ij} are non-negative and so x is an optimal solution. The optimal objective value (minimum cost) is

$$\sum_{(i,j) \in T_2} c_{ij} x_{ij} = 2 \times 3 + 1 \times 2 + 3 \times 2 + 2 \times 2 = 6 + 2 + 6 + 4 = 18.$$

3c

In a general network flow problem, there may or may not be a unique optimal solution.
(i) Suppose the optimal solution is unique. If the supplies b_i are integers, will the flows x_{ij} in the optimal solution be integers?
(ii) Suppose there is more than one optimal solution. If the supplies b_i are integers, will the flows x_{ij} in an optimal solution be integers?
Explain your answers.

Answer: (i) If the supplies b_i are integers, the flows x_{ij} corresponding to any spanning tree, i.e., a basic feasible solution x, will also be integers because we find the flows from the flow balance at each node and, using leaf elimination, the only computations are additions and subtractions. Thus, similarly, the optimal flow solution will consist of integers.

(ii) Not necessarily. By convexity, if x^1 and x^2 are distinct optimal solutions, then $x^3 = (1 - \lambda)x^1 + \lambda x^2$ is also an optimal solution, for any real $\lambda \in (0, 1)$. Thus if the elements of x^1 and x^2 are all integers, the elements of x^3 will not be integers if we choose λ appropriately.

Good luck!