Last time: primal simplex algorithm in matrix form (6.2)

Self-study: chapters 6.3-6.5

- example: see section 6.3 in Vanderbei's book
- dual simplex in matrix form: see section 6.4

Two-step methods in chapter 6.5

Initial dictionary

$$\begin{aligned}\zeta &= \quad\quad c_{\mathcal{N}}^T x_{\mathcal{N}} \\ x_{\mathcal{B}} &= b - N x_{\mathcal{N}}.\end{aligned}$$

Today:

- negative transpose property: proof 6.6
- sensitivity analysis (section 7.1)

Interior point methods 17

Negative transpose property - chapter 6.6

Consider the primal LP problem (P)

$$\max \quad c^T x \quad \text{s.t.} \quad Ax + w = b, \quad x, w \geq O.$$

and the dual (D)

$$\min \quad b^T y \quad \text{s.t.} \quad A^T y - z = c, \quad y, z \geq O.$$

Alternatively: (P) is

$$\max \quad \bar{c}^T \bar{x} \quad \text{s.t.} \quad \bar{A}\bar{x} = \bar{b}, \quad \bar{x} \geq O.$$

and the dual (D)

$$\min \quad \hat{b}^T \hat{y} \quad \text{s.t.} \quad \hat{A}^T \hat{y} = \hat{c}, \quad \hat{y} \geq O.$$

Here

$$\bar{A} = \begin{bmatrix} A & I \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} c \\ O \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} x \\ w \end{bmatrix},$$

and

$$\hat{A} = \begin{bmatrix} -I & A^T \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} O \\ b \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} z \\ y \end{bmatrix},$$

Complementary - primal and dual basis: column j is in basis in $\bar{A}$ if and only if column j is *not* in basis in $\hat{A}$.

In the beginning the m last columns in $\bar{A}$ are in basis, and the n first columns in $\hat{A}$ are in basis.

After a few pivots

$$\bar{A} = \begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} \bar{A}_N & \bar{A}_B \end{bmatrix} P$$

for a permutation matrix P. The columns in $\bar{A}$ are permuted. Since the corresponding pivots occur in the dual

Here

$$\bar{A} = [\; A \quad I \;], \quad \bar{c} = \begin{bmatrix} c \\ O \end{bmatrix}, \quad \bar{x} = \begin{bmatrix} x \\ w \end{bmatrix},$$

and

$$\hat{A} = [\; -I \quad A^T \;], \quad \hat{b} = \begin{bmatrix} O \\ b \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} z \\ y \end{bmatrix},$$

Complementary - primal and dual basis: column j is in basis in $\bar{A}$ if and only if column j is *not* in basis in $\hat{A}$.

In the beginning the m last columns in $\bar{A}$ are in basis, and the n first columns in $\hat{A}$ are in basis.

After a few pivots

$$\bar{A} = [\; A \quad I \;] = [\; \bar{A}_N \quad \bar{A}_B \;]\, P$$

for a permutation matrix P. The columns in $\bar{A}$ are permuted. Since the corresponding pivots occur in the dual

$$\hat{A} = [\; -I \quad A^T \;] = [\; \hat{A}_B \quad \hat{A}_N \;]\, P$$

But $P^{-1} = P^T$, so $PP^T = I$. Which means that

$$\bar{A}\hat{A}^T = [\; \bar{A}_N \quad \bar{A}_B \;]\, PP^T \begin{bmatrix} \hat{A}_B^T \\ \hat{A}_N^T \end{bmatrix} = \bar{A}_N\hat{A}_B^T + \bar{A}_B\hat{A}_N^T$$

and in addition we have that

$$\bar{A}\hat{A}^T = [\; A \quad I \;] \begin{bmatrix} -I \\ A \end{bmatrix} = -A + A = O.$$

So:

$$\bar{A}_N\hat{A}_B^T + \bar{A}_B\hat{A}_N^T = O$$

By some algebra we get that

$$\bar{A}_B^{-1}\bar{A}_N = -(\hat{A}_B^{-1}\hat{A}_N)^T$$

This shows the negative transpose property.

$$\bar{A}_B^{-1}\bar{A}_N\hat{A}_B^T(\hat{A}_B^{-1})^T = -\bar{A}_B^{-1}\bar{A}_B\hat{A}_N^T(\hat{A}_B^{-1})^T$$

$$\hat{A} = \begin{bmatrix} -I & A^T \end{bmatrix} = \begin{bmatrix} \hat{A}_B & \hat{A}_N \end{bmatrix} P$$

But $P^{-1} = P^T$, so $PP^T = I$. Which means that

$$\bar{A}\hat{A}^T = \begin{bmatrix} \bar{A}_N & \bar{A}_B \end{bmatrix} PP^T \begin{bmatrix} \hat{A}_B^T \\ \hat{A}_N^T \end{bmatrix} = \bar{A}_N\hat{A}_B^T + \bar{A}_B\hat{A}_N^T$$

and in addition we have that

$$\bar{A}\hat{A}^T = \begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} -I \\ A \end{bmatrix} = -A + A = O.$$

So:

$$\bar{A}_N\hat{A}_B^T + \bar{A}_B\hat{A}_N^T = O$$

By some algebra we get that

$$\bar{A}_B^{-1}\bar{A}_N = -(\hat{A}_B^{-1}\hat{A}_N)^T$$

This shows the negative transpose property.

Sensitivity analysis

Sensitivity analysis (section 7.1): what happens with the solutions when the parameters are changed?

Look at a question like this for LP: given an optimal basis, how much can each coefficient in the objective function be altered without the present basic solution becoming non optimal?

We can find the answer via duality!

Recall that when A_B is the optimal basis we have

primal basic
dual basic

$$
\begin{aligned}
x_B^* &= A_B^{-1} b, \\
y_N^* &= (A_B^{-1} A_N)^T c_B - c_N, \\
\eta^* &= c_B^T A_B^{-1} b.
\end{aligned}
$$

$x_N^* = 0$
$y_B^* = 0$

$$
\begin{aligned}
\zeta &= \quad c_{\mathcal{N}}^T x_{\mathcal{N}} \\
x_{\mathcal{B}} &= b - N x_{\mathcal{N}}.
\end{aligned}
$$

$$
\begin{aligned}
\eta &= c_B^T A_B^{-1} b + (c_N - (A_B^{-1} A_N)^T c_B)^T x_N \\
x_B &= A_B^{-1} b - A_B^{-1} A_N x_N
\end{aligned}
$$

$$
\begin{aligned}
-\xi &= -c_B^T A_B^{-1} b - (A_B^{-1} b)^T z_B \\
z_N &= (A_B^{-1} A_N)^T c_B - c_N + (A_B^{-1} A_N)^T z_B.
\end{aligned}
$$

So: assume that only c is altered (among the data). Then y_N^* is altered, but not x_B^*. So if c is not altered too much, such that the new vector y_N^* is nonnegative, then x_B^* will still be optimal!

Assume that c is altered to $c + t \cdot \Delta c$, where t is a number and Δc is an "perturbation vector" (often a unity vector). Then y_N^* is altered to $y_N^* + t \cdot \Delta y_N$ where

$$\Delta y_N = (A_B^{-1} A_N)^T \Delta c_B - \Delta c_N.$$

So the present basis will still be optimal (after the perturbation in c) if

$$(*) \quad y_N^* + t \cdot \Delta y_N \geq O.$$

The sensitivity analysis boils down to determining the smallest and the largest value of t so that $(*)$ holds!!

Example:

p 101

$$
\begin{array}{lllll}
\max & 5x_1+ & 4x_2+ & 3x_3 & \\
\text{s.t.} & & & & \\
\text{(i)} & 2x_1+ & 3x_2+ & x_3 & \leq 5 \\
\text{(ii)} & 4x_1+ & x_2+ & 2x_3 & \leq 11 \\
\text{(iii)} & 3x_1+ & 4x_2+ & 2x_3 & \leq 8
\end{array}
$$

$$x_1, x_2, x_3 \geq 0.$$

Optimal dictionary, where $B = \{3, 1, 5\}$ and $N = \{4, 2, 6\}$

$$
\begin{array}{llllllll}
\eta & = 13 & - & x_4 & - & 3x_2 & - & x_6 \\
\hline
x_3 & = 1 & + & 3x_4 & + & x_2 & - & 2x_6 \\
x_1 & = 2 & - & 2x_4 & - & 2x_2 & + & x_6 \\
x_5 & = 1 & + & 2x_4 & + & 5x_2 & &
\end{array}
$$

$-A_B^{-1} A_N$

Note: be aware of the order of the variables in the matrix calculations!

We want to look at a change of the coefficient 3 to x_3 in the obj.func.
Therefore, let $\Delta c_B = (1,0,0)^T$ and $\Delta c_N = (0,0,0)^T$. The matrix $A_B^{-1}A_N$ is found from the optimal dictionary like this:

$$-A_B^{-1}A_N = \begin{bmatrix} 3 & 1 & -2 \\ -2 & -2 & 1 \\ 2 & 5 & 0 \end{bmatrix} \text{ which gives}$$

$$\Delta y_N = (A_B^{-1}A_N)^T \Delta c_B - \Delta c_N =$$

$$\begin{bmatrix} -3 & -1 & 2 \\ 2 & 2 & -1 \\ -2 & -5 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}.$$

$$\begin{array}{rcl} \eta & = & c_B^T A_B^{-1} b + (c_N - (A_B^{-1}A_N)^T c_B)^T x_N \\ \hline x_B & = & A_B^{-1} b - A_B^{-1} A_N x_N \end{array}$$

So: B will be the optimal basis if

$$(*) \quad (y_N^* + t \cdot \Delta y_N)^T = (1,3,1) + t \cdot (-3,-1,2) \geq O$$

$$\text{i.e.} \quad 1 - 3t \geq 0, \; 3 - t \geq 0, \; 1 + 2t \geq 0.$$

This gives $-1/2 \leq t \leq 1/3$. So the coefficient of x_3 (which was 3) can vary between $3 - 1/2 = 5/2$ and $3 + 1/3 = 10/3$.
Finally: note what happens if we use $\Delta c_B = O$!

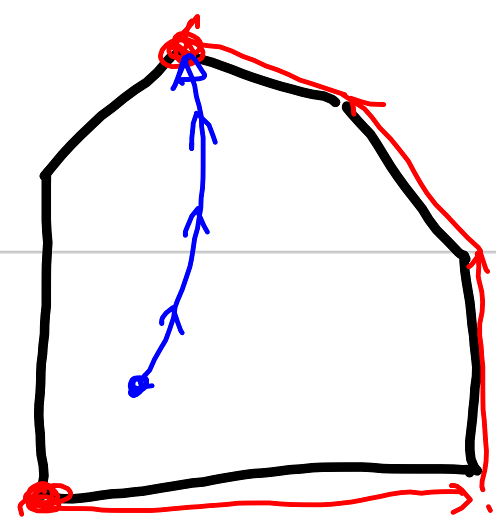
This sensitivity analysis shows how important dictionaries (or the concept of a basis) are to understand linear programming!

Further themes are:

- interior pont methods
- some game theory
- convexity (geometrical aspects of LP), and
- network flow problems.

LP. Kap. 17: Interior-point methods

- the simplex algorithm moves along the boundary of the polyhedron P of feasible solutions
- an alternative is interior-point methods
- they find a path in the interior of P, from a starting point to an optimal solution
- for large-scale problems interior-point methods are usually faster
- we consider the main idea in these methods



1. The barrier problem

Consider the LP problem

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \\ Ax & \leq b, \\ x & \geq O \end{aligned}$$

and its dual

$$\begin{aligned} \min \quad & b^T y \\ \text{s.t.} \quad & \\ A^T y & \geq c, \\ y & \geq O \end{aligned}$$

Introduce slack variables w in the primal and (negative) slack z in the dual, which gives

Primal (P):

$$\begin{array}{lrcrcl} \max & c^T x & & & & \\ \text{s.t.} & & & & & \\ & Ax & + & w & = & b, \\ & & & x, w & \geq & O \end{array}$$

Dual (D):

$$\begin{array}{lrcrcl} \min & b^T y & & & & \\ \text{s.t.} & & & & & \\ & A^T y & - & z & = & c, \\ & & & y, z & \geq & O \end{array}$$

- We want to rewrite the problems such that we eliminate the constraints $x, w \geq O$ og $y, z \geq O$, but still avoid negative values (and 0) on the variables!!
- This is achieved by a logarithmic barrier function, and we get the following *modified* primal problem

The barrier problem:

$$(P_\mu): \quad \begin{array}{ll} \max & c^T x + \mu \sum_j \log x_j + \mu \sum_i \log w_i \\ \text{s.t.} & \\ & Ax + w = b \end{array}$$

- (P_μ) is not equivalent to the original problem (P), but it is an approximation
- it contains a parameter $\mu > 0$.
- remember: $x_j \to 0^+$ implies that $\log x_j \to -\infty$.
- (P_μ) is a nonlinear optimization problem
- interpretation/geometry: see Figure 17.1 in Vanderbei: level curves for f_μ, polyhedron P, central path when $\mu \to 0$.

- Goal: shall see that (P_μ) has a unique optimal solution $x(\mu)$ for each $\mu > 0$, and that $x(\mu) \to x^*$ when $\mu \to 0^+$, where x^* is the unique optimal solution of (P). (Note: w is uniquely determined by x)

The central path