LP. Lecture Game theory

Chapter 11: game theory

- matrix games
- optimal strategies
- von Neumann's minmax theorem
- connection to LP
- useful LP modeling of (certain) minmax and maxmin problems

Example: Paper-Scissors-Rock (= saks-papir-stein) The game:

- Two persons independently choose one of the three options: Paper, Scissors or Rock
- Rules: Paper beats Rock, Rock beats Scissors, Scissors beats Paper.

Payoff matrix:

	r	K	Э	
	0	-1	1	P
A =	1	0	-1	R
	1	1	$egin{array}{c} 1 \\ -1 \\ 0 \end{array}$	S

0 0 0

Row player (R) chooses a row i, the Column player (K) chooses a column j, and the payoff is the entry a_{ij}: the row player pays the column player a_{ij} kroner (NOK).

 Similar for a general m × n matrix A = [aij]; this is called a Matrix game. (two-player, finite, zero-sum)

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- Analyze mathix gaves - Assure both players are intelligent and know the rules

PURE STRATEGIES Consider 5276A = 12201433Consider column-player K . K words to matimize worst case, i.e the smallest payoff V= max min agi (max min strategy) R wants to minimize worst ase, i.e. the maximal payof U* = min max dig (min at shadegy) i=2 and i= 3 is optimal for both! azy is a saddle-point This gave has a value $V = V^* = U^* = 2$

So,

• K wins at least 2 (independent of Rs choice)

· R pays at most 2 (independent of Ks choice)

• If both players play optimally, R pays K 2 (not fair!)

The Rock-scissors-paper gave has no velue: Vy=-1 < 1 = Vy

No particular optimal pure/fixed strategy for any of the players

Randomized strategies may be a better choice

Randomized strategies

- The choice studied above is called a deterministic strategy: choose one row (or column).
- In Paper-Scissors-Rock no deterministic strategy can always win (if the game is played repeatedly), e.g., if R always chooses Paper, soon K will choose Scissors.
- May be better to use a randomized strategy: R chooses row i with probability y_i, and, independently, K chooses column j with probability x_i.

► So:

$$\sum_{i=1}^{m} y_i = 1, \ y_i \ge 0 \ (i \le m)$$
$$\sum_{j=1}^{m} x_j = 1, \ x_j \ge 0 \ (j \le n)$$

The Expected payoff from R to K is (recall probability theory!):

$$\sum_{i}\sum_{j}y_{i}a_{ij}x_{j}=y^{T}Ax$$

What are good strategies for K and R?

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Both players should optimize their expected worst case

Wants minimal guaranteed payoff (to K)

For the paper-scissors-tock game

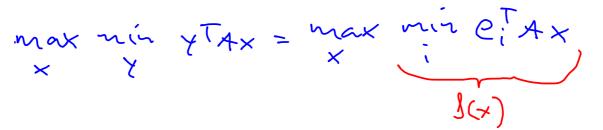
$$x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$
 is optimal momential for k
 $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is optimal minimax for R
(and shows later)

where $e_i = (0,...,0,1,0,...,0)$, the i'th coordinate vector.

Note:

- the minimum value f(x) is in fact the minimal element of the vector Ax (which is again a convex combination of the columns of A, a mix of "pure" payoffs)
- we have reduced a continuous problem to a discrete problem

We may conclude that



Can formulate this as an LP (with the above v=f(x) as the objective function)

max
$$V$$

such that $V \leq e; TAx$ $i = 1, ..., m$
 $\sum x_i = ($
 $x_{7,0}$

K's problem in matrix rotation

max V
suchthat
$$ve - Ax \leq O$$

 $eTx = 1$ (LP-k)
 $x \neq O$

where $e_i = (1,...,1)$

Optimal value

V = max min YTAX

The maxmin problem: strategy for player K

Let e_i be the *i*th coordinate vector and e the all ones vector (of suitable size). Note that an LP with the feasible set being the standard simplex $S = \{y : \sum_i y_i = 1, y \ge 0\}$ is easy, so we get:

$$v^* = \max_{x} \min_{y} y^T A x = \max_{x} \min_{y} e_i^T A x$$

objective value jor Therefore player K's strategy problem may be written as the LP problem

$$\max\{v: v \leq e_i^T A x \ (i \leq m), \sum_j x_j = 1, x \geq O\}$$

with variables $v \in \mathbb{R}$, $x \in \mathbb{R}^n$; or in matrix notation:

$$\begin{array}{ccc} \max & v \\ (\text{LP-K}) & \text{s.t.} & \\ & ve - Ax \leq 0 \\ & e^T x = 1 \\ & x \geq 0 \end{array}$$

Thus: we can find an optimal strategy x for K efficiently by solving this LP. 9/11

The minmax problem: strategy for player R

Similar analysis for player R:

$$u^* = \min_{y} \max_{x} y^T A x = \min_{y} \max_{j} y^T A e_j$$

objecting halve for

So, player R's strategy problem becomes the LP problem

$$\min\{u: u \ge y^T A e_j \ (j \le n), \sum_i y_i = 1, y \ge 0\}$$

with variables $u \in \mathbb{R}$, $y \in \mathbb{R}^m$; which is

$$\begin{array}{ccc} \min & u \\ (\text{LP-R}) & \text{s.t.} & \\ & ue - A^T y \geq 0 \\ & e^T y = 1 \\ & y \geq 0 \end{array}$$

The min-nex theorem So: LP-R is dual to LP-K By duality: if x* and y* are Jeasible for LP-K and LP-R, Then minyTAX* & max Y* AX Yn Cn prinal dich objection objection where where with equality ill x and y & are optical ! optimal golutions xt and y exist since both problems are jeasible (feasible stoclosed and bounded)

This proves the minmax theorem

The minmax theorem

Theorem [John von Neumann(1928)] Let x^* be an optimal strategy for player K and y^* an optimal strategy for player R. Then

$$w^* = \max_{x} (y^*)^T A x = \min_{y} y^T A x^* = u^*$$

i.e., $\min_{y} \max_{x} y^{T} A x = \max_{x} \min_{y} y^{T} A x$.

Proof. One can check that problem LP-R is the dual LP of problem LP-K. (Exercise!) So, by the duality theorem of LP the optimal value v^* of LP-K equals the optimal value u^* of LP-R, and this proves the theorem.

- The common value v* = u* is called the value of the game: this is the expected payoff when both players play optimally
- It is also possible to prove the LP duality theorem from von Neumann's theorem
- Solve the LP's above, for some selected A's, using OPL-CPLEX.

Solving the Rock-Scissors-Paper problem

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Strundnic gave !, so value $V = V = 0$
Lets the $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
Then $A_x = 0$
So min $yTA_x = 0 = Y^{*}$
 $\frac{1}{2}$
By duality:
This is the optical value of LP-k
 $=) x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is optical

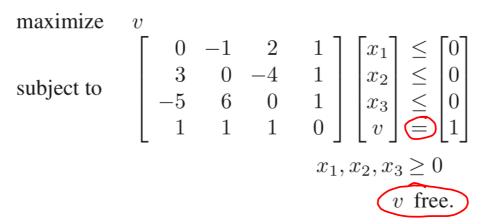
The same is true for R:

Y = (= = = =) is optimal

Solving a modified problem

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -3 & 0 & 4 \\ 5 & -6 & 0 \end{bmatrix}$$

Column player K's problem in matrix form



Non-standard form: equality constraint and a free variable

Write the problem in equation form:

maximize
$$v$$

subject to $-x_2 + 2x_3 + v \le 0$
 $3x_1 - 4x_3 + v \le 0$
 $-5x_1 + 6x_2 + v \le 0$
 $x_1 + x_2 + x_3 = 1$
 $x_1, x_2, x_3 \ge 0$.

Remove equality constraint first:

$$\frac{\xi}{x_4 = -2 + 2x_1 + 3x_2 - v}$$

$$x_5 = 4 - 7x_1 - 4x_2 - v$$

$$x_6 = 5x_1 - 6x_2 - v$$

$$x_3 = 1 - x_1 - x_2$$

v is free, does not belong as non-basic

pivot to make v basic

Reduced dictionary is in standard form!

$$\frac{\xi = -2 + 2x_1 + 3x_2 - x_4}{x_5 = 6 - 9x_1 - 7x_2 + x_4}$$
$$x_6 = 2 + 3x_1 - 9x_2 + x_4$$
$$x_3 = 1 - x_1 - x_2 .$$

Feasible, run simplex algorithm to find optimal solution

$$\frac{102\xi = -16 - 27x_5 - 13x_6 - 62x_4}{102x_1 = 40 - 9x_5 + 7x_6 + 2x_4}$$

$$102x_2 = 36 - 3x_5 - 9x_6 + 12x_4$$

$$102x_3 = 26 + 12x_5 + 2x_6 - 14x_4.$$
Optical Solution $\chi = \frac{1}{102} \left(40, 36, 26 \right)$

$$\chi = \frac{1}{102} \left(62, 27, 13 \right)$$

So the row-player has a slight advantage