LP. Lecture Game theory

Chapter 11: game theory

- matrix games
- optimal strategies
- von Neumann's minmax theorem
- connection to LP
- useful LP modeling of (certain) minmax and maxmin problems

Example: Paper-Scissors-Rock (= saks-papir-stein)

The game:

- Two persons independently choose one of the three options: Paper, Scissors or Rock
- Rules: Paper beats Rock, Rock beats Scissors, Scissors beats Paper.

Payoff matrix:

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} P & R & S \end{array}$$

$$\begin{array}{cc} & \begin{array}{ccc} P & R & S \end{array} \\ A = & \left[\begin{array}{rrr} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{array}\right] \begin{array}{c} P \\ R \\ S \end{array} \end{array}$$

- Row player (R) chooses a row i, the Column player (K) chooses a column j, and the payoff is the entry a_{ij}: *the row player pays the column player a_{ij} kroner (NOK).*
- Similar for a general $m \times n$ matrix $A = [a_{ij}]$; this is called a Matrix game. (two-player, finite, zero-sum)

- Analyze matrix games
- Assume both players are intelligent and know the rules

What is a good strategy?

PURE STRATEGIES

Consider

$$A = \begin{bmatrix} 5 & 2 & 7 & 6 \\ 1 & 2 & 2 & 0 \\ 1 & 4 & 3 & 3 \end{bmatrix}$$

Consider column-player K:

K wants to maximize worst case,
i.e the smallest payoff

$v^* = \max_j \min_i a_{ij}$ (maxmin strategy)

R wants to minimize worst case,
i.e. the maximal payoff

$u^* = \min_i \max_j a_{ij}$ (minmax strategy)

$i=2$ and $j=3$ is optimal for both!

a_{23} is a <u>saddle-point</u>

This game has a <u>value</u>

$V = v^* = u^* = \underline{\underline{2}}$

So,
- K wins at least 2 (independent of Rs choice)
- R pays at most 2 (independent of Ks choice)
- If both players play optimally, R pays K 2 (not fair!)

The Rock-scissors-paper game has no value: $V_* = -1 < 1 = U_*$

No particular optimal pure/fixed strategy for any of the players

Randomized strategies may be a better choice

Randomized strategies

- The choice studied above is called a deterministic strategy: choose one row (or column).
- In Paper-Scissors-Rock no deterministic strategy can always win (if the game is played repeatedly), e.g., if R always chooses Paper, soon K will choose Scissors.
- May be better to use a randomized strategy: R chooses row i with probability y_i, and, *independently*, K chooses column j with probability x_j.
- So:

$$\sum_{i=1}^{m} y_i = 1, \quad y_i \geq 0 \ \ (i \leq m)$$
$$\sum_{j=1}^{m} x_j = 1, \quad x_j \geq 0 \ \ (j \leq n)$$

The Expected payoff from R to K is (recall probability theory!):

$$\sum_i \sum_j y_i a_{ij} x_j = y^T A x$$

What are good strategies for K and R?

Both players should optimize their expected worst case

Optimal strategies

$\boxed{\text{optimize expected payoff } y^T A x}$

K's situation: wants maximal guaranteed payoff (from R)

if K chooses (randomized) strategy $\underline{x}$

the best choice for R would be

$$\min_{y} y^T A \underline{x}$$

Therefore K's best choice is to maximize this:

$$\max_{x} \min_{y} y^T A x$$ (maxmin strategy for K)

Similar analysis for R

Wants minimal guaranteed payoff (to K)

$$\min_{y} \max_{x} y^T A x$$ (minmax strategy for R)

For the paper-scissors-rock game

$x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is optimal maxmin for K

$y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is optimal minmax for R

(will show later)

Lets look at K's problem

$$\max_x \underbrace{\min_y y^T A x}_{f(x)}$$

Inner optimization is "simple": Given x

$$f(x) = \min_y y^T A x = \min_{i=1,\dots,m} e_i^T A x$$

where e_i = (0,...,0,1,0,...,0) , the i'th coordinate vector.

Note:

- the minimum value f(x) is in fact the minimal element of the vector Ax (which is again a convex combination of the columns of A, a mix of "pure" payoffs)
- we have reduced a continuous problem to a discrete problem

We may conclude that

$$\max_x \min_y y^T A x = \max_x \underbrace{\min_i e_i^T A x}_{f(x)}$$

Can formulate this as an LP (with the above v=f(x) as the objective function)

$$\begin{aligned} \max \quad & v \\ \text{such that} \quad & v \le e_i^T A x \qquad i = 1, \dots, m \\ & \textstyle\sum x_i = 1 \\ & x \geq 0 \end{aligned}$$

K's problem in matrix notation

$$\begin{aligned} \max \quad & v \\ \text{such that} \quad & ve - Ax \leq 0 \\ & e^T x = 1 \\ & x \geq 0 \end{aligned} \qquad \text{(LP-K)}$$

where $e_i = (1,\ldots,1)$

Clearly a LP! (not in standard form)

Can find optimal maxmin randomized strategy x^* for K by solving this!

Optimal value

$$v^* = \max_x \min_y y^T A x$$

The maxmin problem: strategy for player K

Let e_i be the ith coordinate vector and e the all ones vector (of suitable size). Note that an LP with the feasible set being the standard simplex $S = \{y : \sum_i y_i = 1, y \geq O\}$ is easy, so we get:

$$v^* = \max_x \min_y y^T A x = \max_x \underbrace{\min_i e_i^T A x}$$

objective value for x

Therefore player K's strategy problem may be written as the LP problem

$$\max\{v : v \leq e_i^T A x \ (i \leq m), \sum_j x_j = 1, x \geq O\}$$

with variables $v \in \mathbb{R}$, $x \in \mathbb{R}^n$; or in matrix notation:

$$\begin{array}{lll} & \max & v \\ \text{(LP-K)} & \text{s.t.} & \\ & & ve - Ax \leq O \\ & & e^T x = 1 \\ & & x \geq O \end{array}$$

Thus: we can find an optimal strategy x for K efficiently by solving this LP.

The minmax problem: strategy for player R

Similar analysis for player R:

$$u^* = \min_y \max_x y^T A x = \min_y \underbrace{\max_j y^T A e_j}$$

objective value for y

So, player R's strategy problem becomes the LP problem

$$\min\{u : u \geq y^T A e_j \ (j \leq n), \sum_i y_i = 1, y \geq O\}$$

with variables $u \in \mathbb{R}$, $y \in \mathbb{R}^m$; which is

$$\text{(LP-R)} \quad \begin{array}{ll} \min & u \\ \text{s.t.} & \\ & ue - A^T y \geq O \\ & e^T y = 1 \\ & y \geq O \end{array}$$

This is the dual of (LP-K)

The min-max theorem

So: LP-R is dual to LP-K

By duality: if x^* and y^* are feasible for LP-K and LP-R, then

$$\underbrace{\min_{y} y^T A x^*}_{\text{primal objective value}} \leq \underbrace{\max_{x} y^{*T} A x}_{\text{dual objective value}}$$

with equality iff x^* and y^* are optimal!

optimal solutions x^* and y^* exist since both problems are feasible (feasible set is closed and bounded)

This proves the minmax theorem

The minmax theorem

Theorem [John von Neumann(1928)] *Let x^* be an optimal strategy for player K and y^* an optimal strategy for player R. Then*

$$v^* = \max_x (y^*)^T A x = \min_y y^T A x^* = u^*$$

i.e., $\min_y \max_x y^T A x = \max_x \min_y y^T A x$.

Proof. One can check that problem LP-R is the dual LP of problem LP-K. (Exercise!) So, by the duality theorem of LP the optimal value v^* of LP-K equals the optimal value u^* of LP-R, and this proves the theorem. □

- The common value $v^* = u^*$ is called the value of the game: this is the expected payoff when both players play optimally
- It is also possible to prove the LP duality theorem from von Neumann's theorem
- Solve the LP's above, for some selected A's, using OPL-CPLEX.

- If $v^* = u^* = 0$, the game is fair
- symmetric games ($a_{ij} = -a_{ji}$) are fair

Solving the Rock-Scissors-Paper problem

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Symmetric game! so value $v^* = u^* = 0$

Lets try $x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Then $Ax = 0$

So $\min_{y} y^T A x = 0 = v^*$

By duality:

This is the optimal value of LP-k

$\Rightarrow x = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is optimal

The same is true for R:

$y = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is optimal

Solving a modified problem

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -3 & 0 & 4 \\ 5 & -6 & 0 \end{bmatrix}$$

Column player K's problem in matrix form

$$\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & \begin{bmatrix} 0 & -1 & 2 & 1 \\ 3 & 0 & -4 & 1 \\ -5 & 6 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v \end{bmatrix} \begin{matrix} \le \\ \le \\ \le \\ = \end{matrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ & x_1, x_2, x_3 \ge 0 \\ & v \text{ free.} \end{array}$$

Non-standard form: equality constraint and a free variable

Write the problem in equation form:

$$\begin{array}{lrcl} \text{maximize} & v & & \\ \text{subject to} & -x_2 + 2x_3 + v & \le & 0 \\ & 3x_1 \qquad - 4x_3 + v & \le & 0 \\ & -5x_1 + 6x_2 \qquad + v & \le & 0 \\ & x_1 + x_2 + x_3 & = & 1 \\ & x_1, x_2, x_3 & \ge & 0\,. \end{array}$$

Remove equality constraint first:

Eliminate x_3 from equations

$x_3 = 1 - x_1 - x_2$

$$\begin{array}{lrcl} \text{maximize} & v & & \\ \text{subject to} & -2x_1 - 3x_2 + v & \leq & -2 \\ & 7x_1 + 4x_2 + v & \leq & 4 \\ & -5x_1 + 6x_2 + v & \leq & 0 \\ & x_1 + x_2 & \leq & 1 \\ & x_1,\ x_2 & \geq & 0\,. \end{array}$$

write in dictionary form, with x_3 as slack-variable

$$\begin{array}{rcl} \xi & = & v \\ \hline x_4 & = & -2 + 2x_1 + 3x_2 - v \\ x_5 & = & 4 - 7x_1 - 4x_2 - v \\ x_6 & = & 5x_1 - 6x_2 - v \\ x_3 & = & 1 - x_1 - x_2\,. \end{array}$$

v is free, does not belong as non-basic

pivot to make v basic

$$\begin{array}{rcl} \xi & = & -2 + 2x_1 + 3x_2 - x_4 \\ \hline v & = & -2 + 2x_1 + 3x_2 - x_4 \\ x_5 & = & 6 - 9x_1 - 7x_2 + x_4 \\ x_6 & = & 2 + 3x_1 - 9x_2 + x_4 \\ x_3 & = & 1 - x_1 - x_2\,. \end{array}$$

not needed

Reduced dictionary is in standard form!

$$\begin{array}{rcl} \xi & = & -2 + 2x_1 + 3x_2 - x_4 \\ \hline x_5 & = & 6 - 9x_1 - 7x_2 + x_4 \\ x_6 & = & 2 + 3x_1 - 9x_2 + x_4 \\ x_3 & = & 1 - x_1 - x_2 . \end{array}$$

Feasible, run simplex algorithm to find optimal solution

$$\begin{array}{rcl} 102\xi & = & -16 - 27x_5 - 13x_6 - 62x_4 \\ \hline 102x_1 & = & 40 - 9x_5 + 7x_6 + 2x_4 \\ 102x_2 & = & 36 - 3x_5 - 9x_6 + 12x_4 \\ 102x_3 & = & 26 + 12x_5 + 2x_6 - 14x_4 . \end{array}$$

Optimal solution $x^* = \frac{1}{102}(40, 36, 26)$

$y^* = \frac{1}{102}(62, 27, 13)$

value $= v^* = u^* = \frac{-16}{102}$

So the row-player has a slight advantage