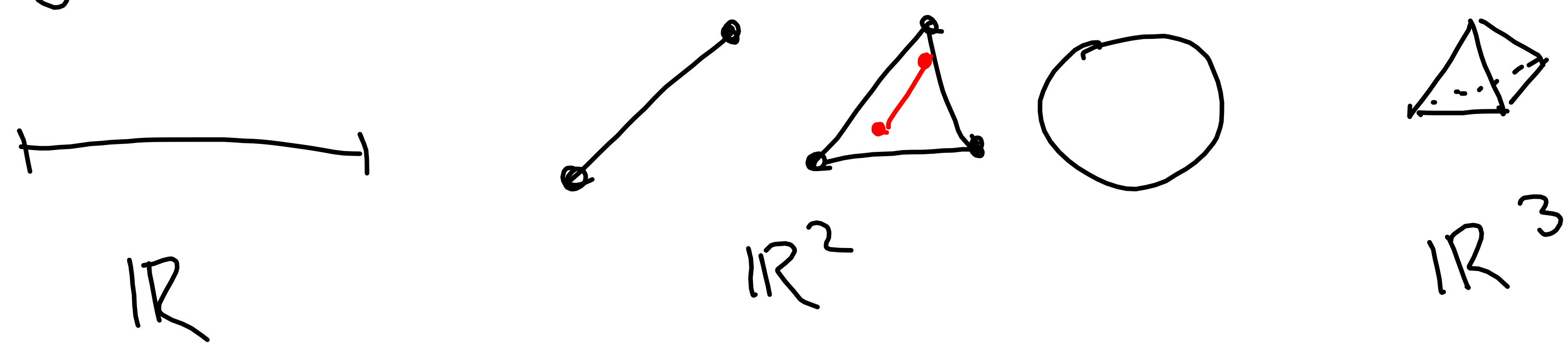
Lecture 10 - Convex Analysis

A set $C \subseteq \mathbb{R}^n$ is <u>convex</u> if

$$(1-\lambda)x_1 + \lambda x_2 \in C$$

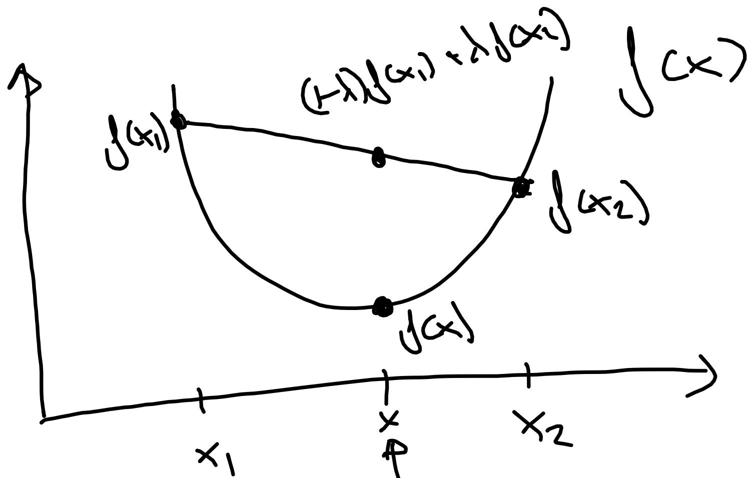
for all $x_1, x_2 \in C$ and $\lambda \in [0,1]$



A function $f : C \to \mathbb{R}$ on a convex set C is <u>convex on C</u> if

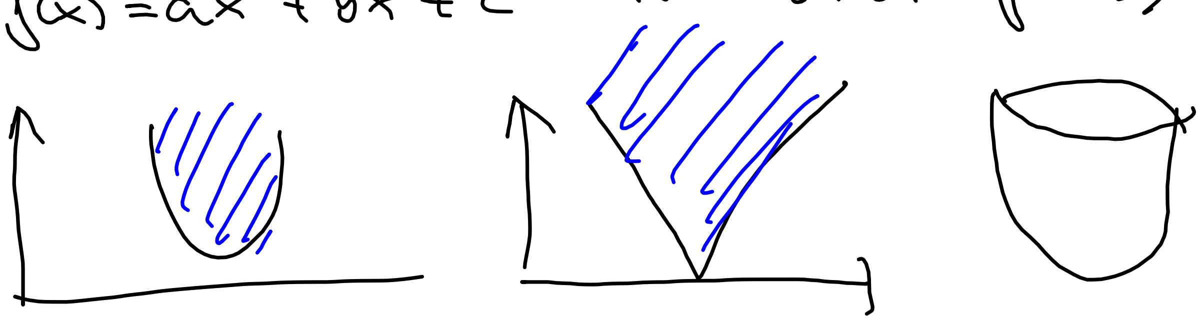
$$f\big((1-\lambda)x_1 + \lambda x_2\big) \le (1-\lambda) f(x_1) + \lambda f(x_2)$$

for all $x_1, x_2 \in C$ and $\lambda \in [0,1]$



$x = (1-\lambda)x_1 + \lambda x_2 \qquad \lambda \in [0,1]$

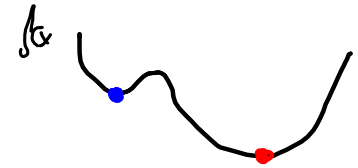
$f(x) = ax^2 + bx + c$ is convex if $a \geq 0$



In optimization we typically want

$$\min_{x \in A} f(x) \quad \text{where } f : \mathbb{R}^n \to \mathbb{R}$$

A global minimum:



x^* such that

$$f(x^*) \leq f(x) \quad \forall x \in A$$

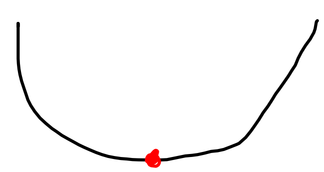
A local minimum

x^* such that

$$f(x^*) \leq f(x) \quad \forall x \in A \text{ „close" to } x^*$$

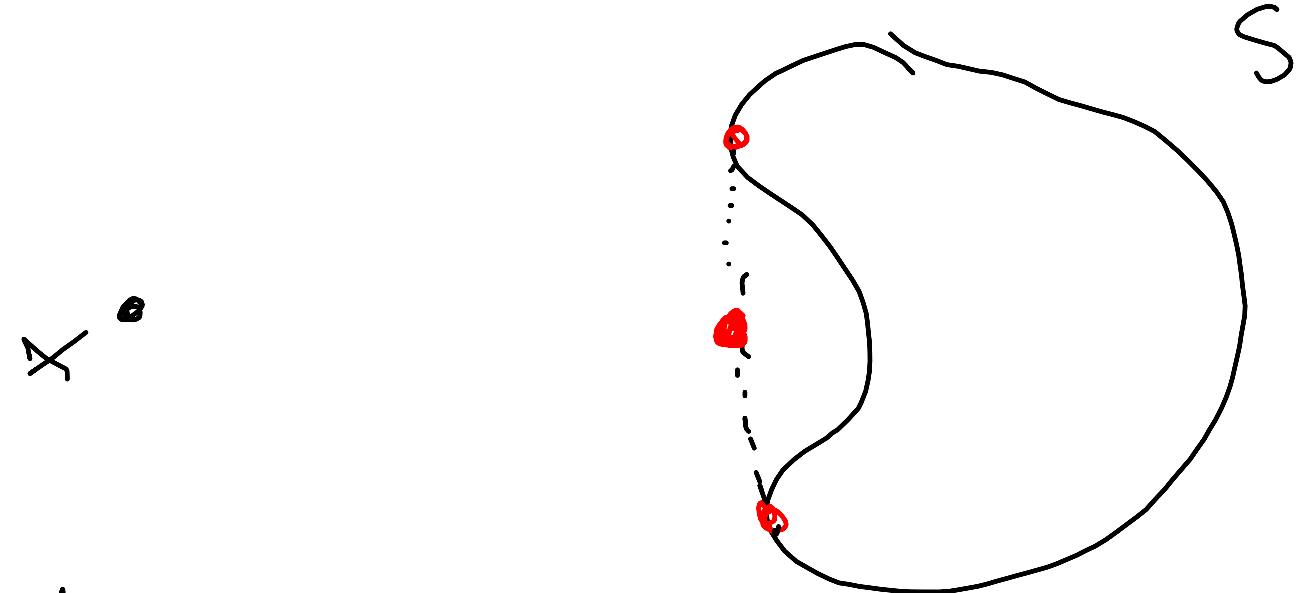
Global minima can be difficult to find

Special case: if f is convex any local minimum is also a global minimum



Example: Closest points

Consider some set $S \subseteq \mathbb{R}^n$, and a point $x \in \mathbb{R}^n$, we would like to find its closest point in S



Special case: if S is convex the closest point is unique

A polyhedron $P \subseteq \mathbb{R}^n$ is defined as

$$P = \{ x \in \mathbb{R}^n : Ax \leq b \}$$

for $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$

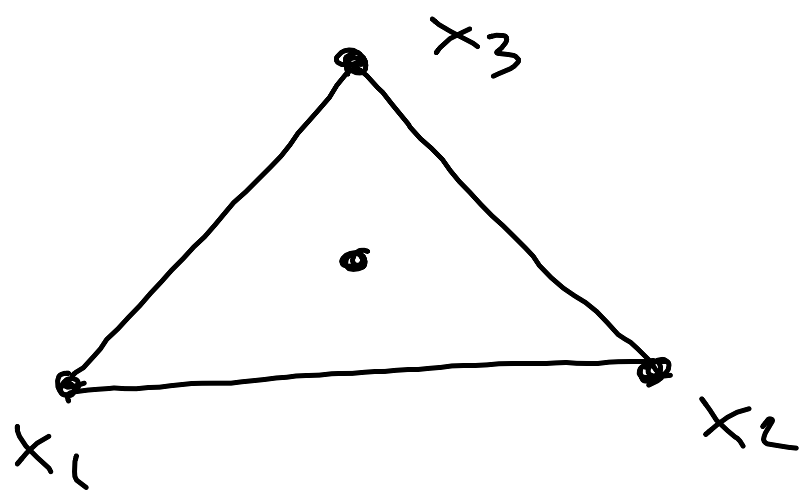
(more generally: the solutions of linear inequalities and equalities)

A <u>convex combination</u> of a set of m points $x_i \in \mathbb{R}^n$ is defined as

$$\sum_{i=1}^{m} \lambda_i x_i$$

for $\sum_{i=1}^{m} \lambda_j = 1$ and $\lambda_j \geq 0$

<u>Example</u>



Proposition 1: Every Polyhedron is a convex set.

Proof: Suppose $x_1, x_2 \in P$, i.e

$$Ax_1 \leq b \quad \text{and} \quad Ax_2 \leq b$$

Consider $x = (1-\lambda)x_1 + \lambda x_2 \quad \lambda \in [0,1]$

check if $x \in P$

$$Ax = A \cdot \left((1-\lambda)x_1 + \lambda x_2\right)$$

$$= (1-\lambda)Ax_1 + \lambda Ax_2$$

$$\leq (1-\lambda)b + \lambda b = b$$

$\Rightarrow Ax \leq b$, so $x \in P$

Therefore P is convex

$\square$

LP

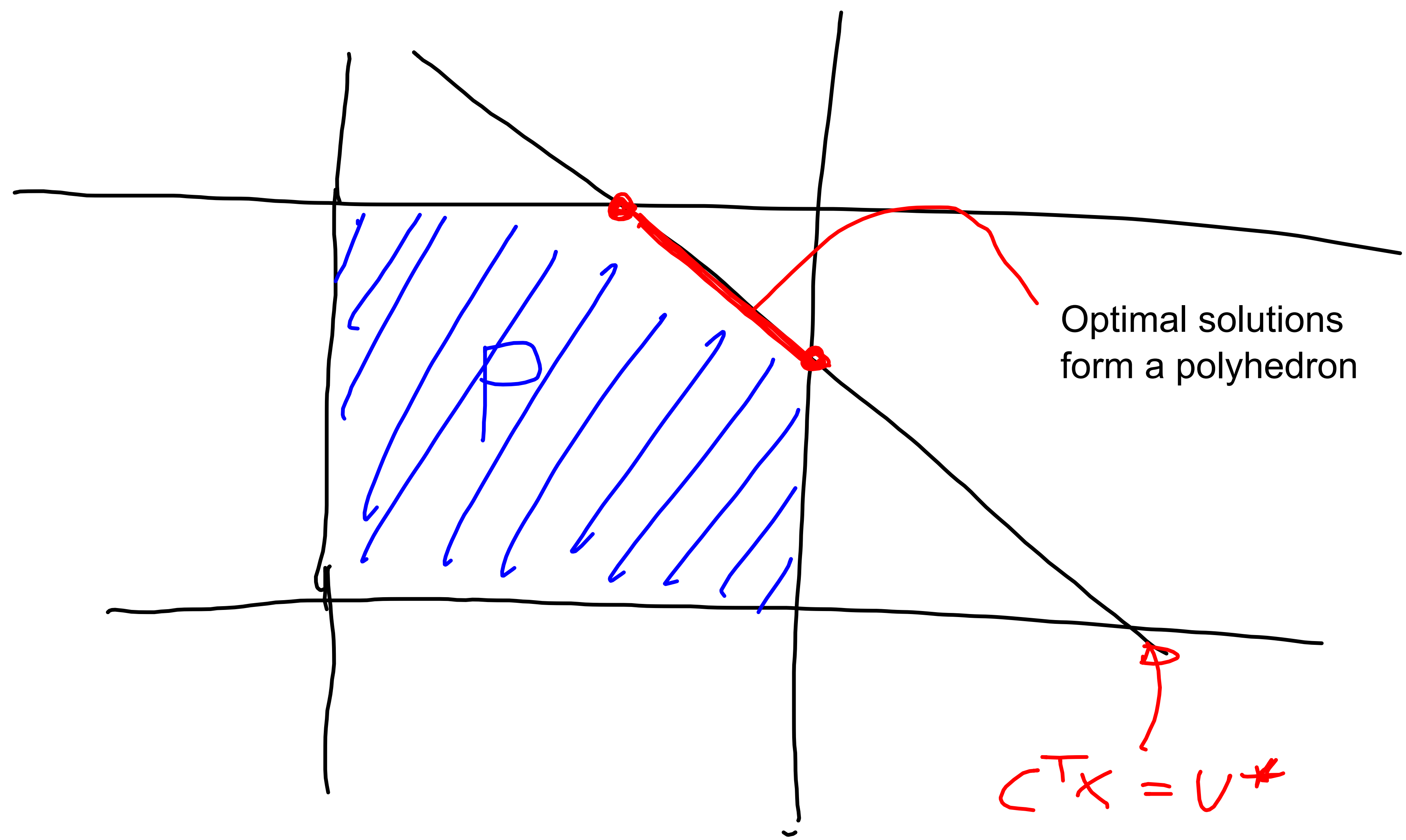
Consider $\max \; c^T x$

$$\text{s.t. } \; Ax \leq b \qquad \text{(LP)}$$
$$x \geq 0$$

for $A \in \mathbb{R}^{m,n}$ and $b \in \mathbb{R}^m$

The feasible set of LP is a polyhedron! And hence convex

So: if x_1, x_2 are feasible, so is $(1-\lambda)x_1 + \lambda x_2$ for any $0 \leq \lambda \leq 1$


P
Optimal solutions
form a polyhedron
$c^T x = v^*$

Proposition 2: If an LP has a finite optimal value, the set of optimal solutions is a polyhedron

Proof: Let v^* denote the optimal value. Then

$$P^* = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0, c^T x = v^*\}$$

is a polyhedron

$\square$

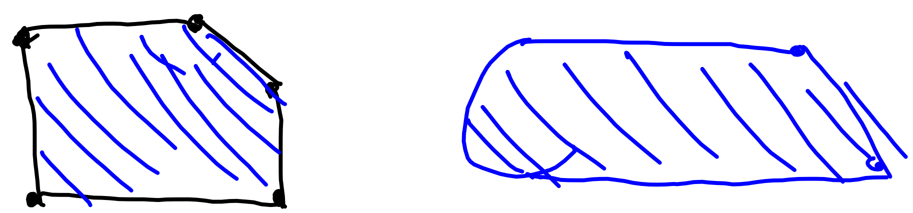
Convex Hulls

Let $S \subseteq \mathbb{R}^n$. Then $\operatorname{Conv}(S)$ denotes the set of all convex combinations of points in S

$$\operatorname{Conv}(S) = \left\{ \sum \lambda_j s_j : s_j \in S, \ \sum \lambda_j = 1, \ \lambda_j \geq 0 \right\}$$

This is called the <u>convex hull</u> of S

<u>Example</u>



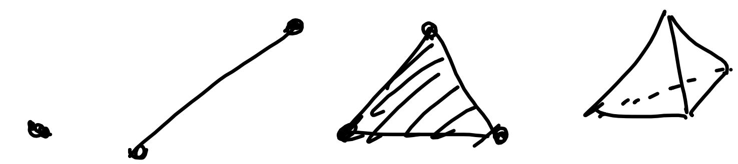
proposition 3: Let $S \subseteq \mathbb{R}^n$. Then $\operatorname{conv}(S)$ is equal to the intersection of <u>all</u> convex sets containing S. Thus $\operatorname{conv}(S)$ is the smallest convex set containing S

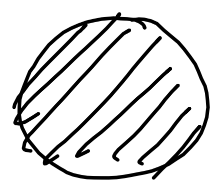
$\square$

Polytopes

A set $P \subseteq \mathbb{R}^n$ is a polytope if it is the convex hull of a finite set of points.

Ex





? Not a polytope

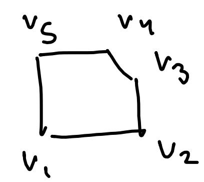
polytopes are bounded, in fact
A set $P \subseteq \mathbb{R}^n$ is a polytope iff it is a bounded polyhedron.

Important for LP: Feasible sets are polyhedrons, polytopes if bounded. In that case the convex hull of a finite set of „vertices".

LP's from another perspective

Consider polytope $P = \mathrm{conv}\left(\{v_1, \dots, v_t\}\right)$
and the optimization problem

$$\max \quad c^T x$$
$$\text{s.t} \quad x \in P$$



This is an LP! with P as the feasible set

So, any feasible x is in P and

$$x = \sum_{j=1}^{t} \lambda_j v_j \quad \text{for } \lambda_j \geq \quad \sum_{j=1}^{t} \lambda_j = 1$$

Let $v^* = \max_j c^T v_j$. then

$$c^T x = c^T \cdot \left(\sum_{j=1}^{t} \lambda_j v_j \right) = \sum_{j=1}^{t} \lambda_j c^T v_j \leq \sum \lambda_j v^*$$
$$= v^*$$

So v^* is an upper bound for the problem, and the set of optimal solutions

$$\mathrm{conv}\left(\{ v_j : c^T v_j = v^* \} \right)$$

is a polytope.

Farkas Lemma: preview

Lemma 4. *Let* $A \in \mathbb{R}^{m,n}$ *and* $b \in \mathbb{R}^m$. *Then the linear system* $Ax \leq b$ *has at least one solution* x *if and only if*

$y^T b \geq 0$ *for every* $y \in \mathbb{R}^m$ *satisfying* $y^T A = O$ *and* $y \geq O$.

Equality constraints and free variables

Consider:

$$\begin{aligned} \max c^T x \\ Ax &= b \\ x &\geq 0 \end{aligned}$$

Rewrite equality constraints as pairs of inequalities:

$$\begin{aligned} \max c^T x \\ Ax &\leq b \\ -Ax &\leq -b \\ x &\geq 0 \end{aligned}$$

Put into block-matrix form:

$$\begin{aligned} \max c^T x & & \\ \begin{bmatrix} A \\ -A \end{bmatrix} x & \begin{matrix} \leq \\ \leq \end{matrix} & \begin{bmatrix} b \\ -b \end{bmatrix} \\ x & \geq & 0 \end{aligned}$$

Dual is:

$$\begin{aligned} \min \begin{bmatrix} b \\ -b \end{bmatrix}^T \begin{bmatrix} y^+ \\ y^- \end{bmatrix} & & \\ \begin{bmatrix} A^T & -A^T \end{bmatrix} \begin{bmatrix} y^+ \\ y^- \end{bmatrix} & \geq & c \\ y^+, y^- & \geq & 0 \end{aligned}$$

Which is equivalent to:

$$\begin{array}{rcl} \min b^T(y^+ - y^-) & & \\ A^T(y^+ - y^-) & \geq & c \\ y^+, y^- & \geq & 0 \end{array}$$

Finally, letting $y = y^+ - y^-$, we get

$$\begin{array}{rcl} \min b^T y & & \\ A^T y & \geq & c \\ y & & \text{free.} \end{array}$$

Moral:

- Equality constraints $\Longrightarrow$ free variables in dual.
- Inequality constraints $\Longrightarrow$ nonnegative variables in dual.

Corollary:

- Free variables $\Longrightarrow$ equality constraints in dual.
- Nonnegative variables $\Longrightarrow$ inequality constraints in dual.

Free dual variables

Consider

$$\max \ c^T x \qquad (LP)$$
$$\text{s.t. } Ax \leq b$$

(no constraints on x)

Take $x = x^+ - x^-$ with $x^+, x^- \geq 0$

Then (LP) can be written

$$\max \ c^T(x^+ - x^-)$$
$$\text{s.t. } A(x^+ - x^-) \leq b$$
$$x^+, x^- \geq 0$$

(Standard form)

The dual is

$$\min \ b^T y$$
$$\text{s.t. } \begin{bmatrix} A^T \\ -A^T \end{bmatrix} y \geq \begin{bmatrix} c \\ -c \end{bmatrix}$$
$$\bar{A} \qquad \geq \qquad \bar{c}$$
$$y \geq 0$$

i.e. $A^T y \geq c$
$-A^T y \geq -c$

i.e. $A^T y = c$
$y \geq 0$

So we end up with an equality constraint in the dual