Den franske spionen løste 30.000 ligninger for å bygge operahuset i Sydney. Nå er han død.

Bertony håndskrev 30.000 ulike ligninger for å finne frem til konstruksjonen av en bue som kunne bære de svært gjenkjennelige hvite «seilene» under konstruksjonen av bygningen.

– Kalkulasjonene ble gjennomgått av den eneste datamaskinen i Australia på den tiden som hadde kapasitet nok til å kunne utføre den type utregninger. Det ble ikke funnet en eneste feil, sier direktør Louise Herron i operahuset.

Lecture 11

Last time: Convex Analysis (Geir Dahls note)

- Convexity, convex hulls etc
- Duality for non-standard problems

Today:

- Recap
- More Convex Analysis (Geir Dahls note)
- Network Flow Problems (V14 & GDs notes)

Duality for non-standard problems

Primal (P)	maximize	minimize	Dual (D)
constraints	$a\,x = b_i$ $a\,x \le b$ $a\,x \ge b_i$	y_i unrestricted $y_i \ge 0$ $y_i \le 0$	variables
variables	$x_j \ge 0$ $x_j \le 0$ unrestricted	$a^T y \ge c_j$ $a^T y \le c_j$ $a^T y = c_j$	constraints

Recap: weak and strong duality

THEOREM 5.1. *If* $(x_1, x_2, \ldots, x_n)$ *is feasible for the primal and* $(y_1, y_2, \ldots, y_m)$ *is feasible for the dual, then*

$$\sum_j c_j x_j \leq \sum_i b_i y_i.$$

THEOREM 5.2. *If the primal problem has an optimal solution,*

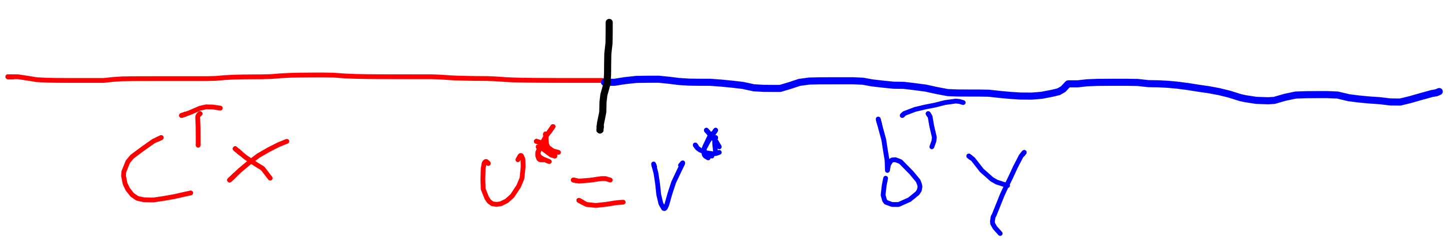
$$x^* = (x_1^*, x_2^*, \ldots, x_n^*),$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \ldots, y_m^*),$$

such that

$$\sum_j c_j x_j^* = \sum_i b_i y_i^*. \tag{5.2}$$



Farkas Lemma

Lemma 4: $Ax \leq b$ has at least one solution if and only if $b^T y \geq 0$ for any $y \geq 0$ such that $A^T y = 0$ $\square$

<u>proof</u>; Formulate as LP

primal problem

$$\max \quad 0^T x = 0$$
$$\text{s.t} \quad Ax \leq b$$
$$x \text{ free}$$

dual problem

$$\min \quad b^T y$$
$$\text{s.t.} \quad A^T y = 0$$
$$y \geq 0$$

$y = 0$ is feasible!

(1) Suppose $b^T y \geq 0$ for all feasible y. Then (D) is bounded and hence has an optimal solution, and so does (P) by duality.

(2) Suppose (P) is feasible (has ~~opt~~ optimal sol) Then $v^* = 0 \leq b^T y$ for all feasible y by weak duality.

6 - Main Theorem for Polyhedra

Theorem 5. *Let $P \subseteq \mathbb{R}^n$ be a nonempty polyhedron. Then there are vectors $v_1, v_2, \ldots, v_s \in \mathbb{R}^n$ and $w_1, w_2, \ldots, w_t \in \mathbb{R}^n$ such that P consists precisely of the vectors x that may be written*

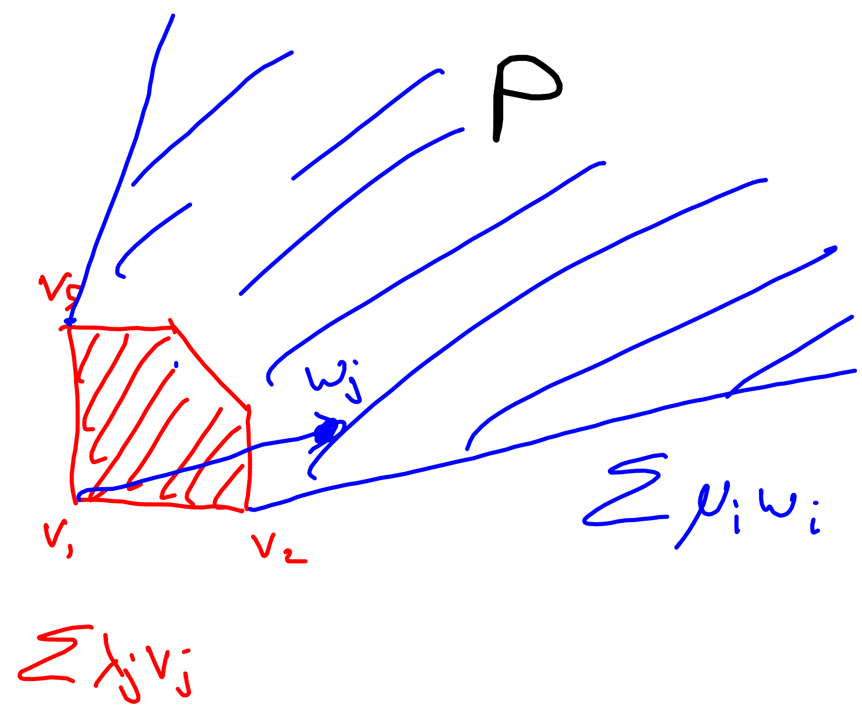
$$x = \underbrace{\sum_{i=1}^{s} \lambda_i v_i}_{\text{convex comb}} + \sum_{j=1}^{t} \mu_j w_j \qquad (6)$$

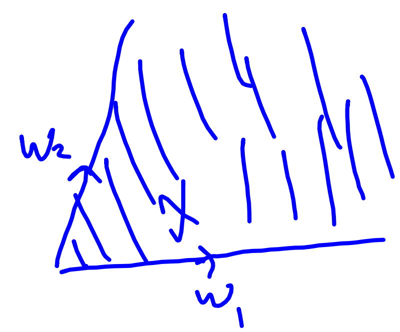
conic combination

where all λ_i and μ_j are nonnegative and $\sum_{i=1}^{s} \lambda_i = 1$.

Conversely, let $v_1, v_2, \ldots, v_s \in \mathbb{R}^n$ and $w_1, w_2, \ldots, w_t \in \mathbb{R}^n$. Then the set Q of vectors x of the form (6) is a polyhedron, so there is a matrix $A \in \mathbb{R}^{m,n}$ and a vector $b \in \mathbb{R}^m$ such that

$$Q = \{x \in \mathbb{R}^n : Ax \leq b\}.$$





Corollary 6: $P \subseteq \mathbb{R}^n$ is a polytope iff it is a bounded polyhedron (no „conic" part, ν's $= 0$)

Application to LP

Consider

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned} \qquad (LP)$$

Feasible set is a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$$

Then the thm says that for $x \in P$

$$x = \sum \lambda_j v_j + \sum \mu_i w_i$$

If $c^T w_j > 0$ for some w_i, (LP) is <u>unbounded</u>

Otherwise it is bounded and thus has an optimal solution.

Extreme points

The points (V_j) defining P

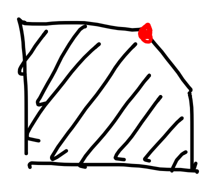
can be choosen to be extreme points of P

That is $x \in P$ for which x can

not be expressed as

$$x = \frac{1}{2}x_1 + \frac{1}{2}x_2 \qquad \text{for } x_1, x_2 \in P$$

unless $x_1 = x_2 = x$



Proposition 7: Let $A \in \mathbb{R}^{m,n}$ have row-rank m $(n \geq m)$

$\boxed{A}$

and consider the polyhedron

$$P = \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$$

and $x \in P$.

Then x is a basic solution in the LP sense iff it is extreme.

Example: $P = \{x \in \mathbb{R}^n : a_1 x_1 + \cdots + a_n x_n = b, x \geq 0\}$

for $a_i > 0$ $\quad b \neq 0$

$A = [a_1, \ldots, a_n]$ has rank 1

Basic solutions have one variable $x_j > 0$
the rest $= 0$

Choose x_j to be basic

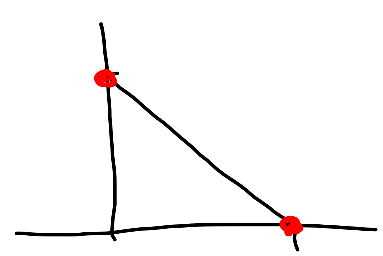
$B = \{j\}$

Find $A_B x_B = b$ with $x_B \geq 0$

for $A_B = [a_j]$

solution is $x_B = \frac{b}{a_j}$

$\Rightarrow x = (0, \ldots, 0, \frac{b}{a_j}, 0, \ldots, 0)$ is a basic solution and hence an extreme point.



Ex 2 : $P = \{x \in \mathbb{R}^4 : 2x_1 + 3x_2 = 3,\ x_1 + x_2 + x_3 + x_4 = 1,\ x \geq 0\}$

$$A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Rank 2, so basic solutions have two nonzero variables.

possible basic ~~sets~~ sets : $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}$

check out $B = \{1,2\}$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}}_{A_B} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x_B} = \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{b}$$

Solution is $x_1 = 0$ and $x_2 = 1$

$x_3 = x_4 = 0$

$x = (0,1,0,0)$ is basic and hence extremal

Fourier-Motzkin Elimination

Computational method to solve

$$Ax \leq b$$

similar to Gaussian elimination

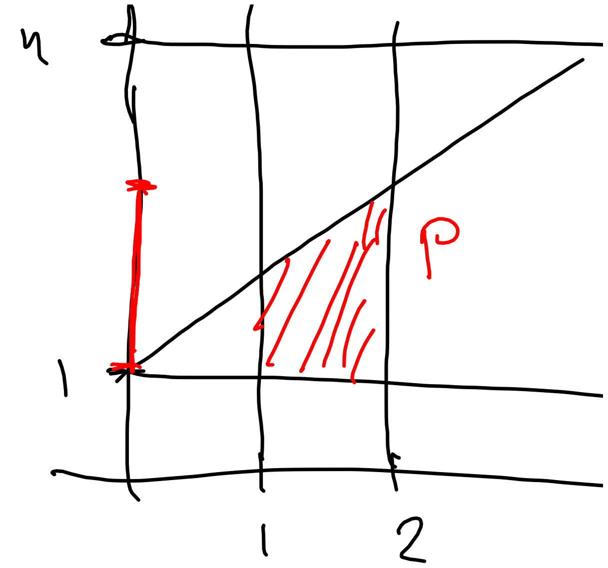
Idea: eliminate one variable at the time, to obtain simpler equations.

Example:

$$1 \leq x_1 \leq 2$$

$$1 \leq x_2 \leq 4$$

$$x_1 - x_2 \geq -1$$



First „solve" for x_1:

$$\left.\begin{array}{r} 1 \leq x_1 \leq 2 \\ x_2 - 1 \leq x_1 \end{array}\right\} \Rightarrow$$

x_1 must satisfy

$$\max(1, x_2 - 1) \leq x_1 \leq 2$$

(when other variables are fixed)

$$1 \leq x_2 \leq 4$$

Eliminate x_1 to get a reduced system

$$1 \leq 2$$

$$x_2 - 1 \leq 2 \Rightarrow x_2 \leq 3$$

$$1 \leq x_2 \leq 4$$

$$\Rightarrow \left.\begin{array}{r} 1 \leq x_2 \leq 3 \\ \max(1, x_2 - 1) \leq x_1 \leq 2 \end{array}\right\} \text{Solution!}$$

To explain the method we assume that we want to eliminate the variables in the order $x_1, x_2, \ldots, x_n$ (although any order will do). The system $Ax \le b$ may be split into three subsystems

$$\begin{array}{llllll} a_{i1}x_1 & +a_{i2}x_2+ & \cdots & +a_{in}x_n & \le b_i & (i \in I^+) \\ 0 \cdot x_1 & +a_{i2}x_2+ & \cdots & +a_{in}x_n & \le b_i & (i \in I^0) \\ a_{i1}x_1 & +a_{i2}x_2+ & \cdots & +a_{in}x_n & \le b_i & (i \in I^-) \end{array} \tag{7}$$

where $I^+ = \{i : a_{i1} > 0\}$, $I^0 = \{i : a_{i1} = 0\}$ and $I^- = \{i : a_{i1} < 0\}$. This system is clearly equivalent to

„Solve" for x_1

$$\begin{array}{rl} a'_{k2}x_2 + \cdots + a'_{kn}x_n - b'_k \le x_1 \le b'_i - a'_{i2}x_2 - \cdots - a'_{in}x_n & (i \in I^+, k \in I^-) \\ a_{i2}x_2 + \cdots + a_{in}x_n \le b_i & (i \in I^0) \end{array} \tag{8}$$

where $b'_i = b_i/|a_{i1}|$ and $a'_{ij} = a_{ij}/|a_{i1}|$ for each $i \in I^+ \cup I^-$.

It follows that $x_1, x_2, \ldots, x_n$ is a solution of the original system (7) if and only if $x_2, x_3, \ldots, x_n$ satisfy

Remaining inequalities

$$\begin{array}{rl} a'_{k2}x_2 + \cdots + a'_{kn}x_n - b'_k \le b'_i - a'_{i2}x_2 - \cdots - a'_{in}x_n & (i \in I^+, k \in I^-) \\ a_{i2}x_2 + \cdots + a_{in}x_n \le b_i & (i \in I^0) \end{array} \tag{9}$$

and x_1 satisfies

$$\max_{k \in I^-}(a'_{k2}x_2 + \cdots + a'_{kn}x_n - b'_k) \le x_1 \le \min_{i \in I^+}(b'_i - a'_{i2}x_2 - \cdots - a'_{in}x_n). \tag{10}$$

x_1 must satisfy this (given $x_2, \ldots, x_n$)

Theorem 8. *The Fourier-Motzkin elimination method is a finite algorithm that finds a general solution to a given linear system $Ax \leq b$. If there is no solution, the method determines this fact by finding an implied and inconsistent inequality ($0 \leq -1$).*

Moreover, the method finds the projection P' of the given polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ into the space of a subset of the variables, and shows that P' is also polyhedron by finding a linear inequality description of P'.