Lecture 13

Last time:

- Convex Analysis (Geir Dahls note)

Today:

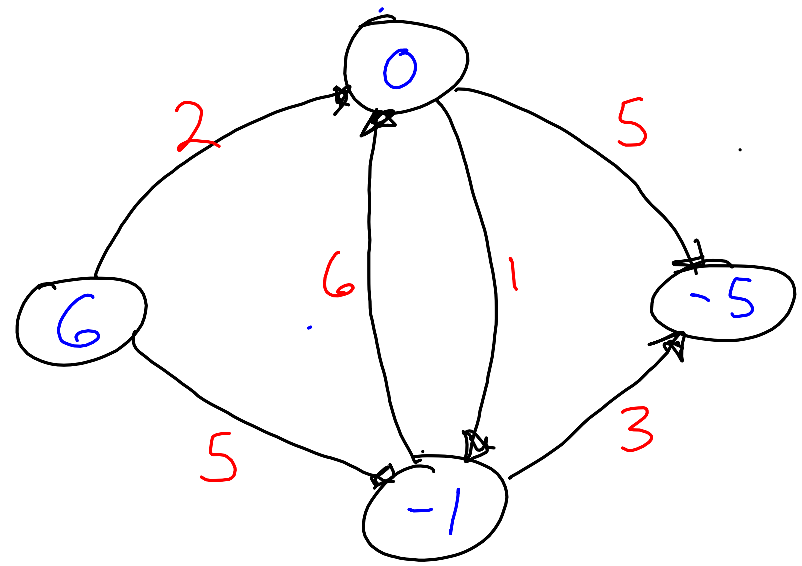
- Network Flow Problems (V14 & GDs notes)

Network Flow Problems

Given directed graph $D = (V, E)$

V – vertices (nodes)

E – edges (arcs)



with

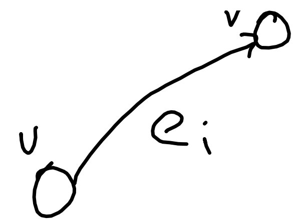
– supply/demand for some materials in the vertices

– transportation costs along edges

How to ship material optimally and so that all supply/demand is satisfied?

Can be formulated as LP's!

Notation



V – vertices $v_1, v_2, \ldots, v_m$

E – directed edges $e_1, e_2, \ldots, e_n$ where
$e_i = (u, v)$ for $u, v \in V$

b – supply (positive) / demand (negative)
b_v for $v \in V$

c – edge costs c_{uv} to ship one
unit along edge $(u,v) \in E$

x – flow, quantity x_{uv} ship along
edge $(u,v) \in E$

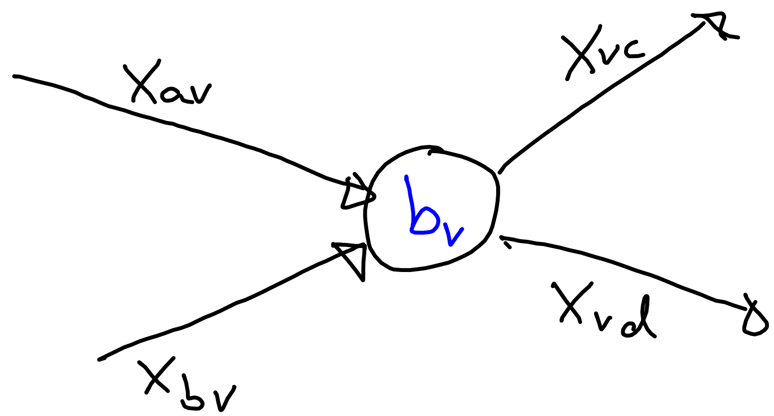
Objective: minimize shipment costs

$$\sum_{(u,v) \in E} x_{uv} \cdot c_{uv}$$

while satisfying constraints

Constraints

Consider $v \in V$



$$\text{inflow}(v) - \text{outflow}(v) = -b_v$$

More formally

$$\sum_{(u,v)\in E} x_{uv} - \sum_{(v,u)\in E} x_{vu} = -b_v \quad \text{for } v \in V$$

Flow balance equations

In addition we require

$$x_{uv} \geq 0 \quad \text{for } (u,v) \in E$$

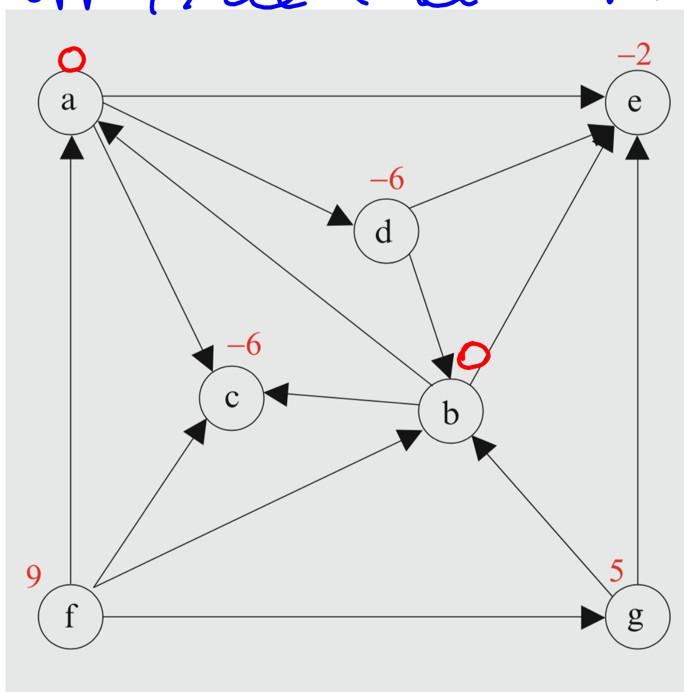
PRIMAL MCF PROBLEM

Matrix notation

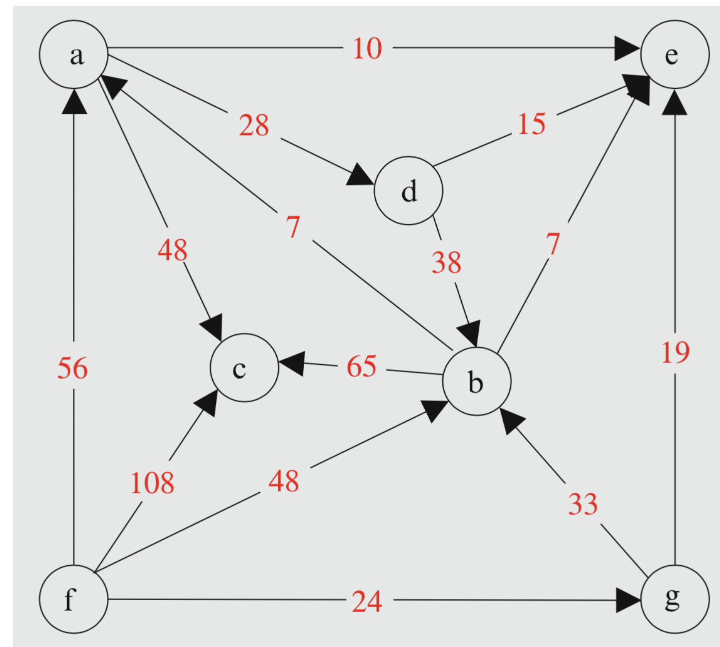
$$\boxed{\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = -b \\ & x \geq 0 \end{aligned}}$$

Example

Supply/demand b_v's



costs c_{uv}'s



n edges →

$$x^T = \begin{bmatrix} x_{\mathrm{ac}} & x_{\mathrm{ad}} & x_{\mathrm{ae}} & x_{\mathrm{ba}} & x_{\mathrm{bc}} & x_{\mathrm{be}} & x_{\mathrm{db}} & x_{\mathrm{de}} & x_{\mathrm{fa}} & x_{\mathrm{fb}} & x_{\mathrm{fc}} & x_{\mathrm{fg}} & x_{\mathrm{gb}} & x_{\mathrm{ge}} \end{bmatrix},$$

m vertices ↓

$$A = \begin{bmatrix} -1 & -1 & -1 & 1 & & & & & 1 & & & & & \\ & & & -1 & -1 & -1 & 1 & & & 1 & & & 1 & \\ 1 & & & & 1 & & & & & & 1 & & & \\ & 1 & & & & & -1 & -1 & & & & & & \\ & & 1 & & & 1 & & 1 & & & & & & 1 \\ & & & & & & & & -1 & -1 & -1 & -1 & & \\ & & & & & & & & & & & 1 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ -6 \\ -6 \\ -2 \\ 9 \\ 5 \end{bmatrix}$$

$$c^T = \begin{bmatrix} 48 & 28 & 10 & 7 & 65 & 7 & 38 & 15 & 56 & 48 & 108 & 24 & 33 & 19 \end{bmatrix}.$$

Node-edge incidence matrix

Duality

primal : $\min\ c^T x$

s.t. $Ax = -b$

$x \geq 0$

(no slack variables)

Dual : $\max\ -b^T y$

s.t. $A^T y \leq c$

y free

Introduce slack variables z

$$\boxed{\begin{array}{ll} \max & -b^T y \\ \text{s.t} & A^T y + z = c \\ & z \geq 0 \\ & y \text{ free} \end{array}}$$

(dual MCF)

Network notation:

$$\max\ - \sum_{v \in V} b_v \cdot y_v$$

s.t. $y_v - y_u + z_{uv} = c_{uv}$ for $(u,v) \in E$

$z_{uv} \geq 0$

y is free

So: one constraint per edge

Complementary slack

If x, y, z are feasible and optimal

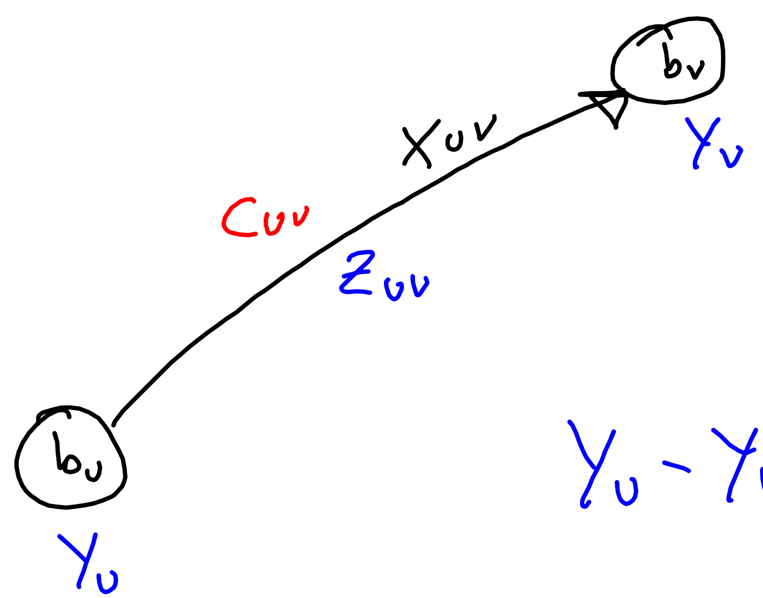
$$x_{uv} \cdot z_{uv} = 0$$

i.e. one of them must be zero

Equivalently: if

$$y_v < y_u + c_{uv}, \quad ,$$

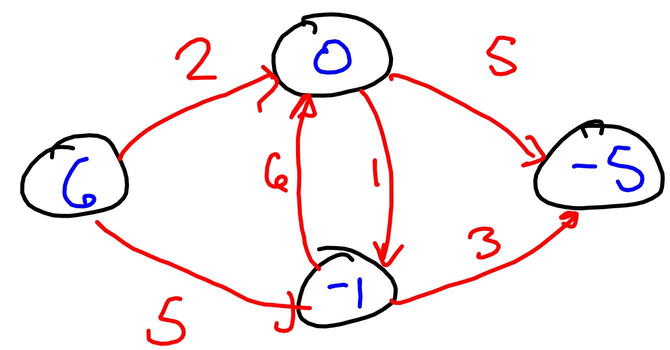
then $z_{uv} > 0$ and $x_{uv} = 0$



$$y_u - y_v + z_{uv} = c_{uv}$$

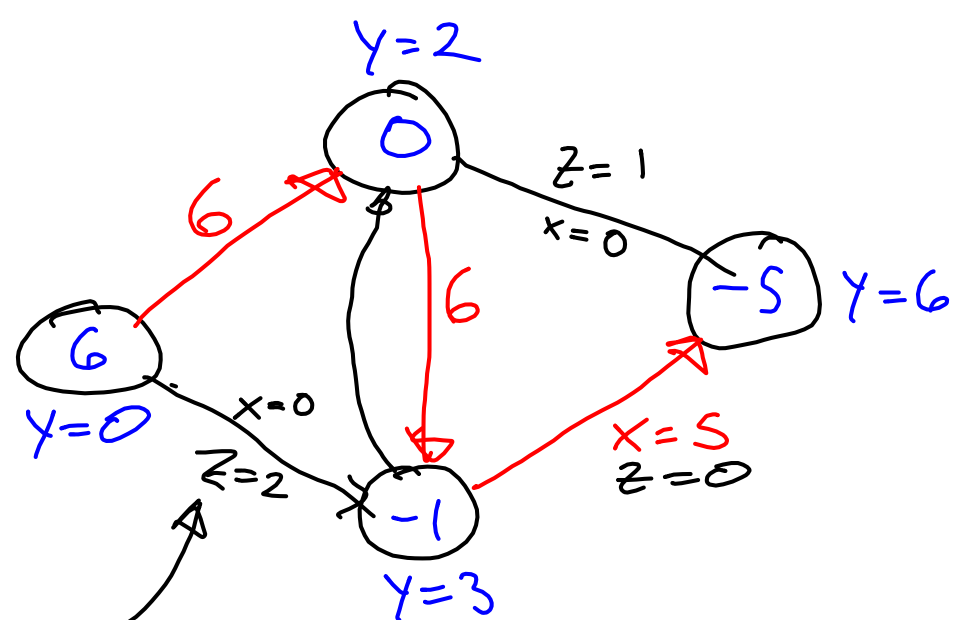
Simple example

Demand & cost



Flow

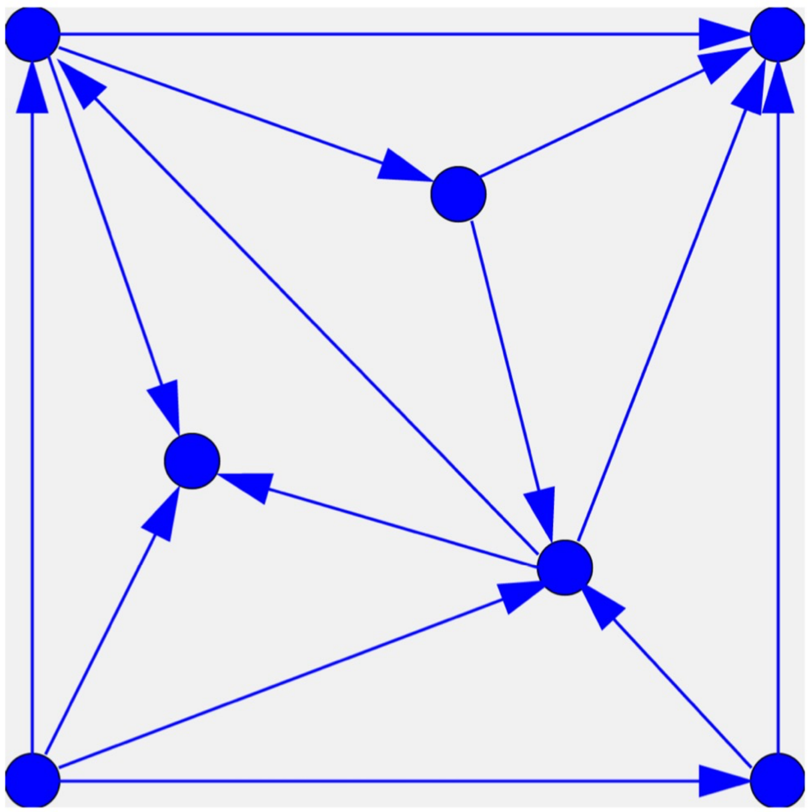
1. choose edges
2. compute x
3. compute y
4. compute z



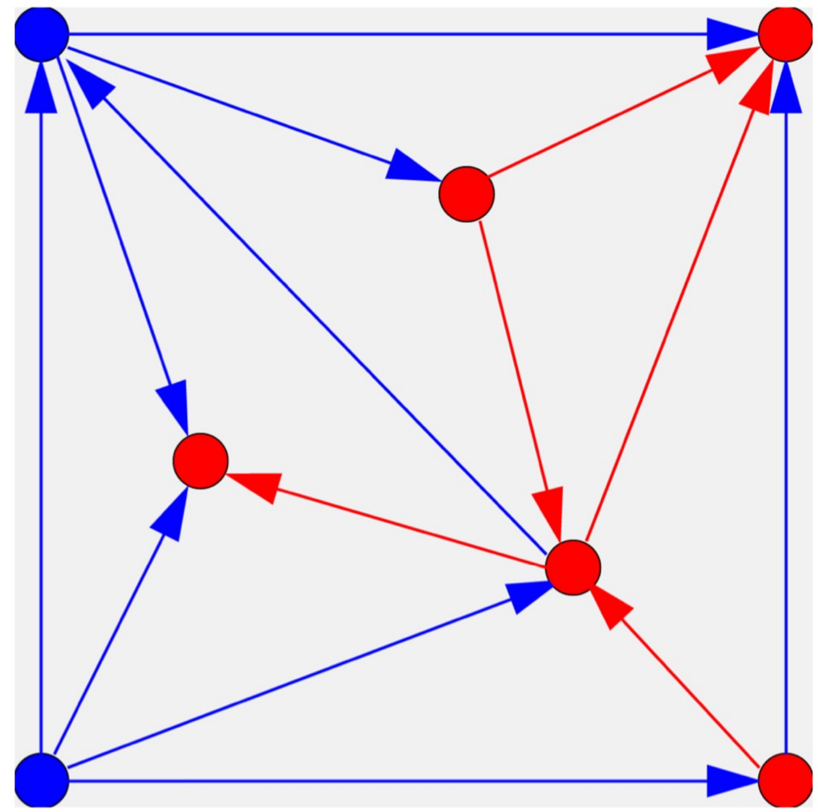
$3-0+z=5$

Optimal solution!

Subnetworks



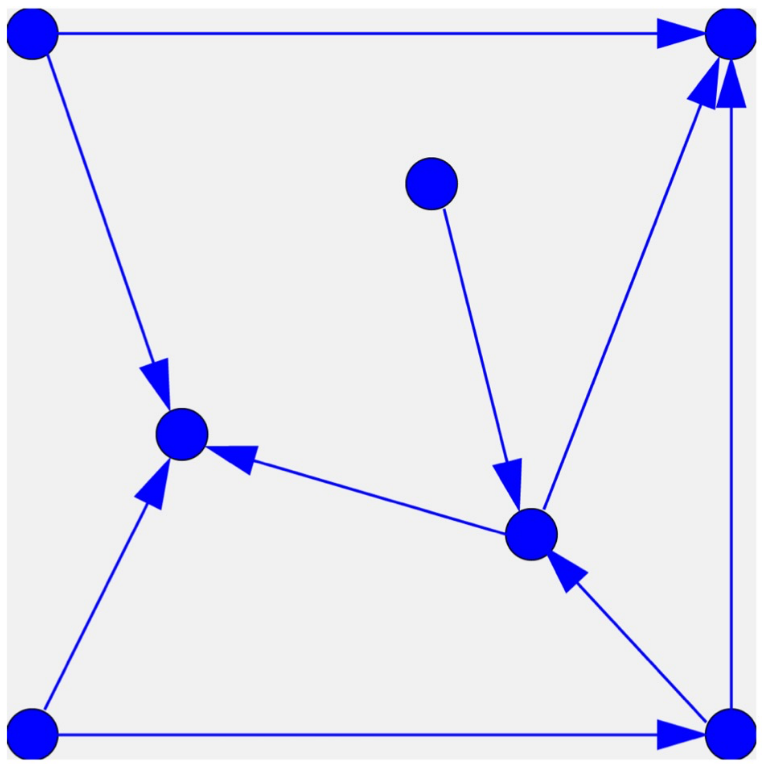
Network



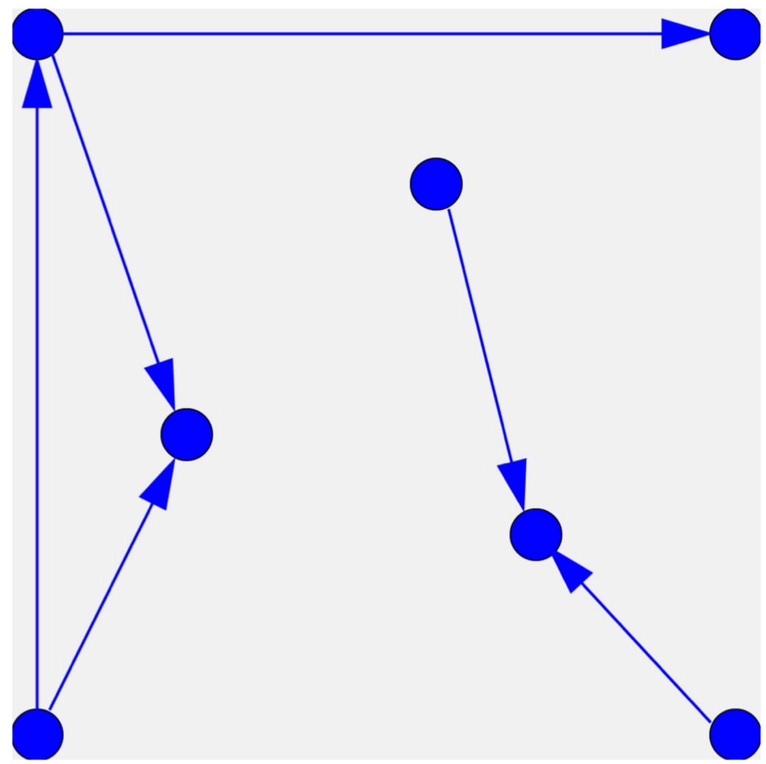
Subnetwork

Connectedness

Graph connected if there is a path between each pair of vertices

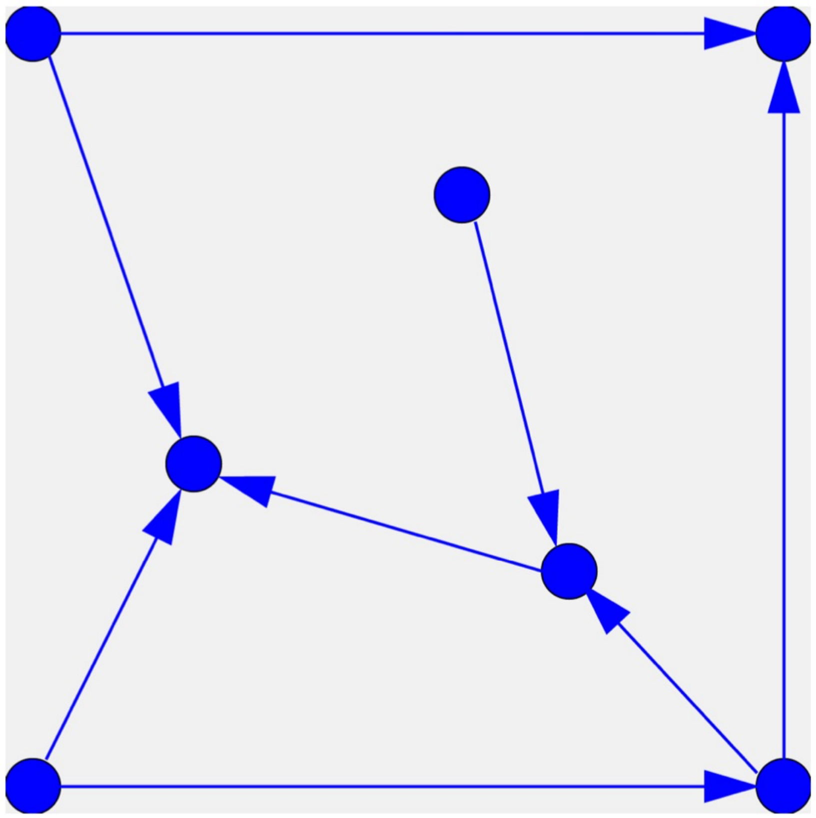


Connected

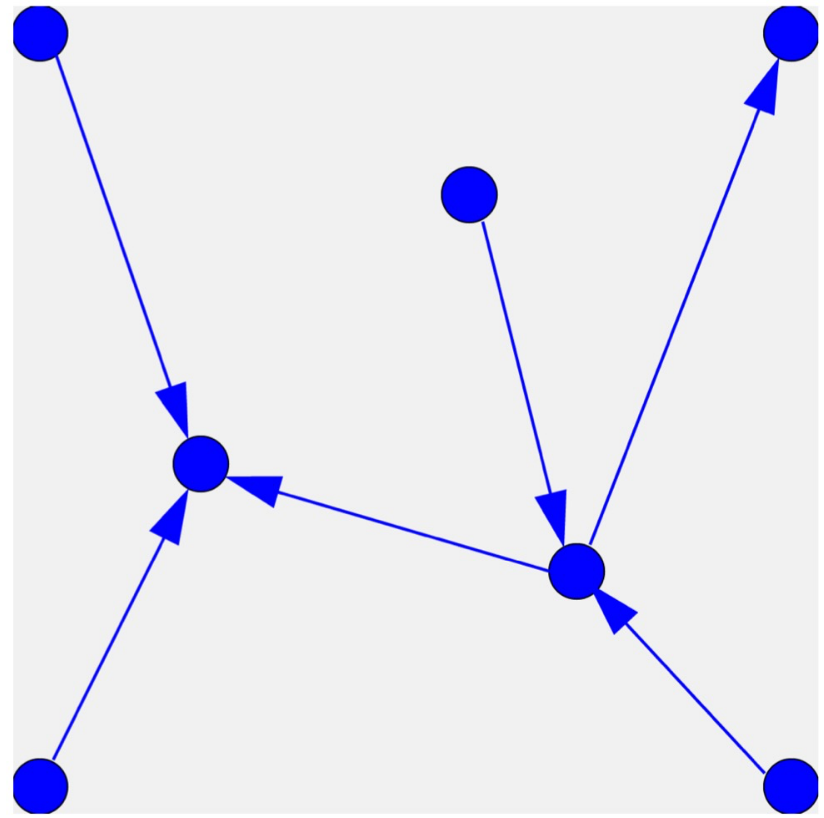


Disconnected

Cycles

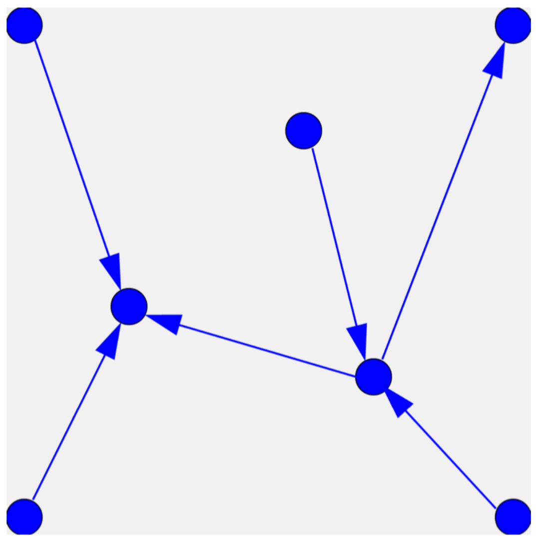


Cyclic

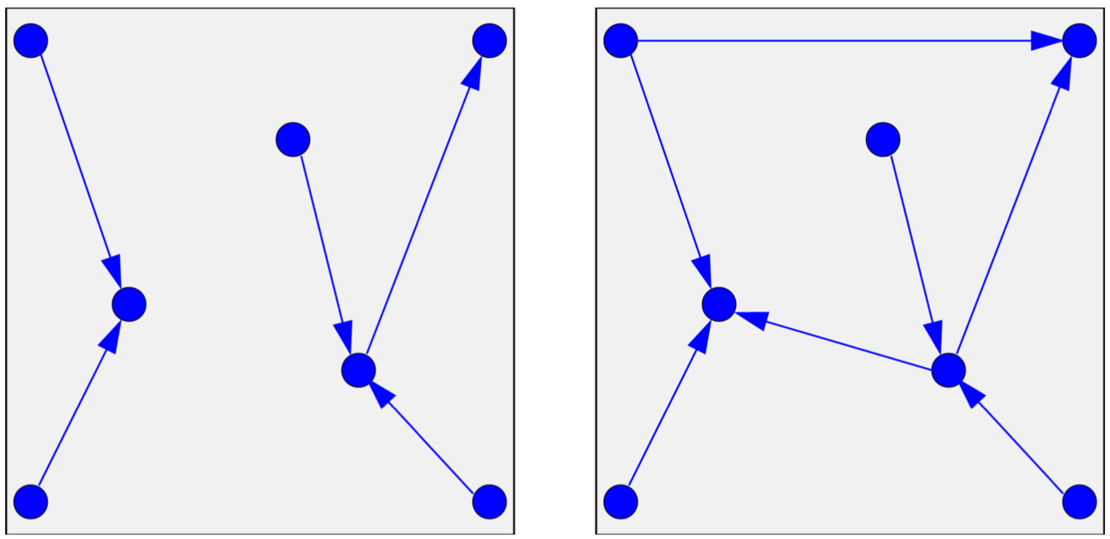


Acyclic

Trees

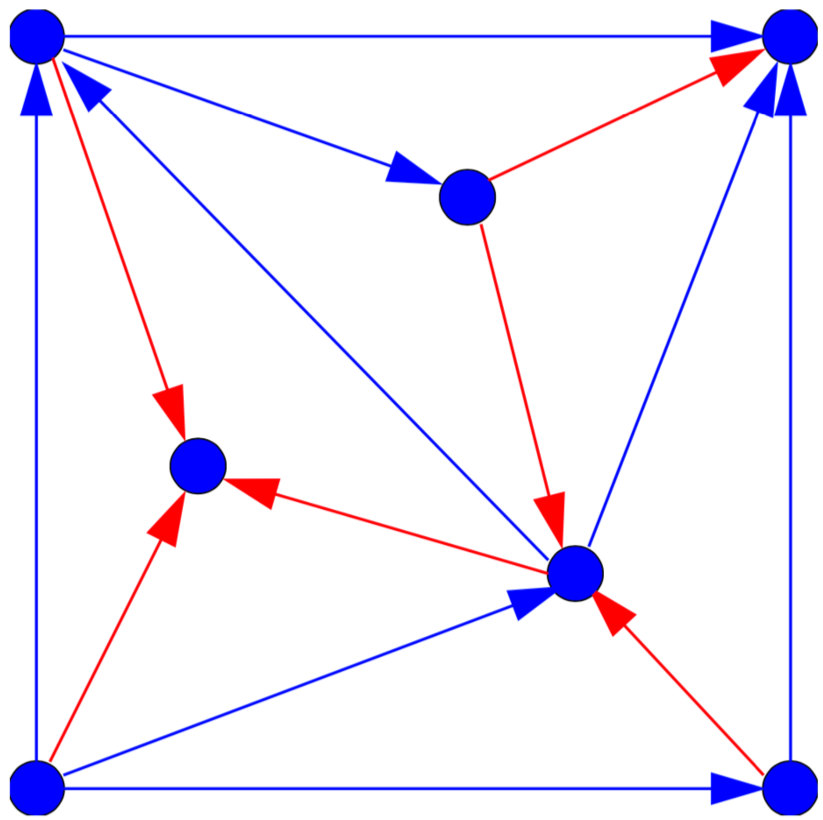


Tree = Connected + Acyclic



Not Trees

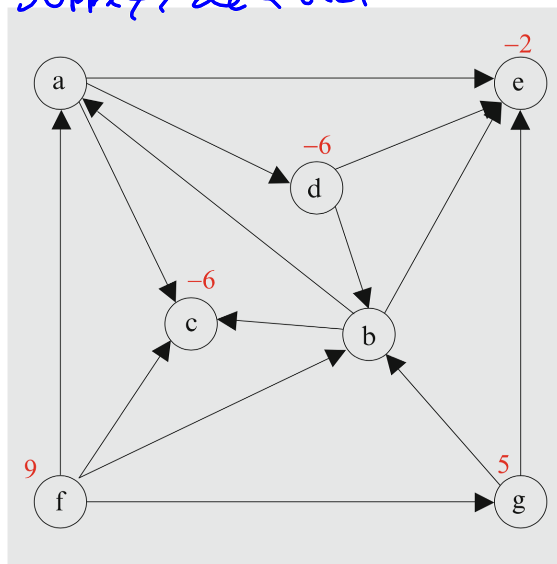
Spanning trees



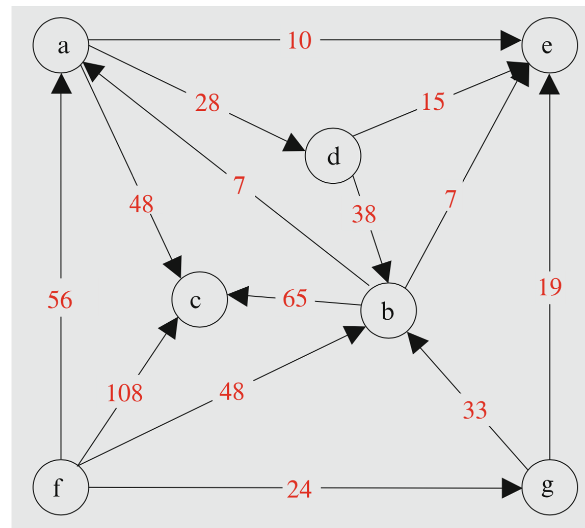
Spanning Tree – A tree touching every node

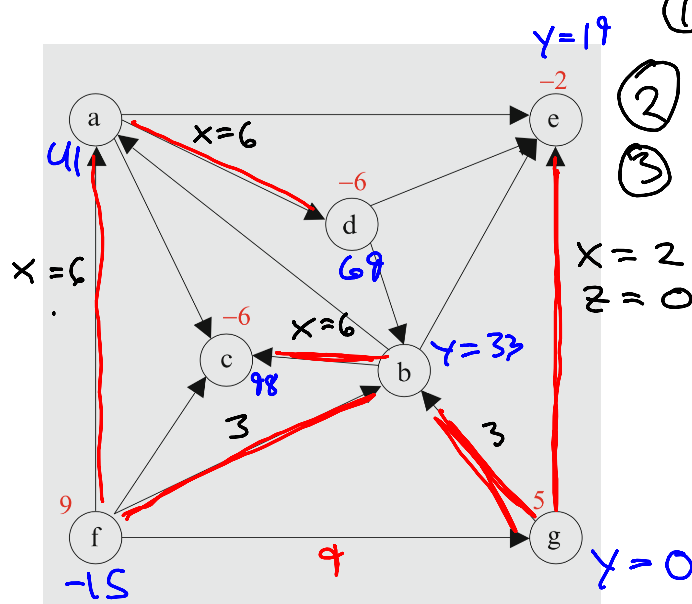
Tree solutions

Supply / demand



Costs





① choose spanning tree

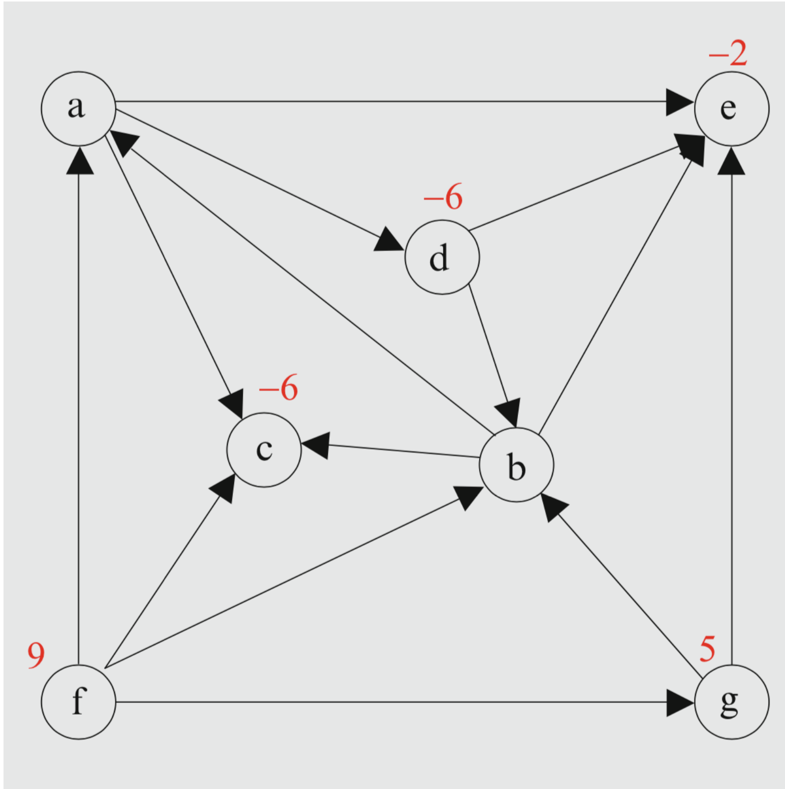
② compute X

③ compute Y

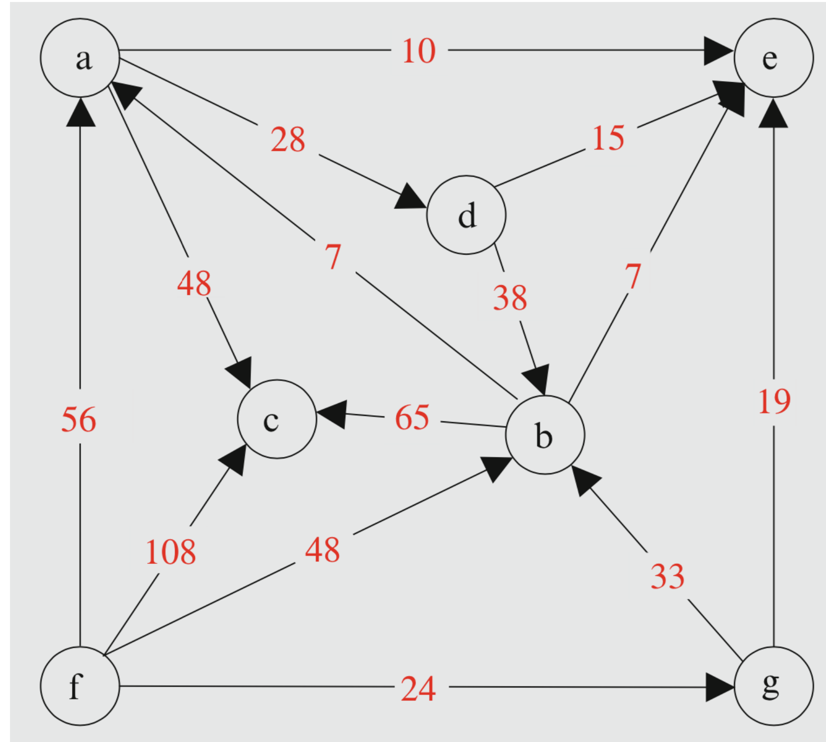
④ compute Z

Tree-edges: $Y_v - Y_u = C_{uv}$

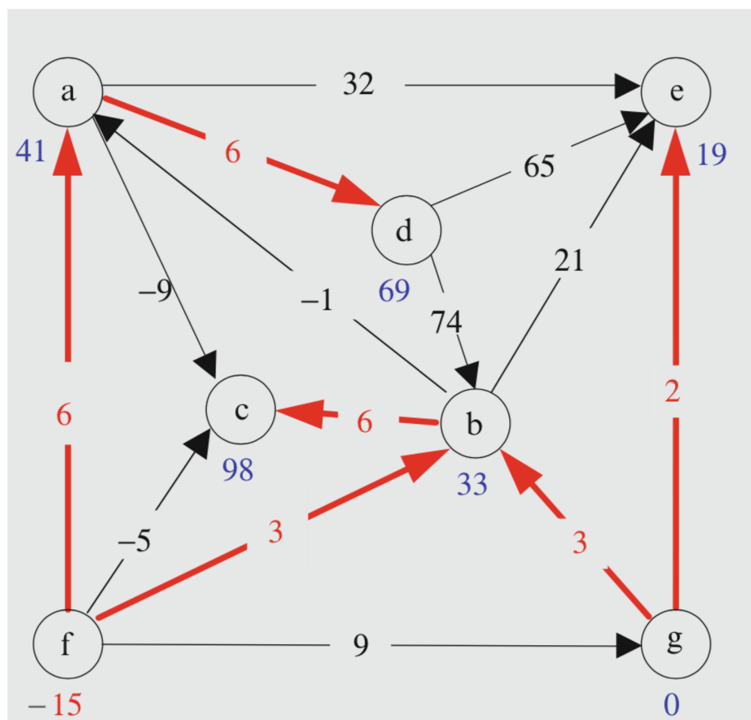
Non-tree-edges: $Y_v - Y_u + Z_{uv} = C_{uv}$



a
e
d
c
b
f
g
−2
−6
−6
9
5



a
e
d
c
b
f
g
10
28
15
48
7
38
7
56
65
19
108
48
33
24



a
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d
c
b
f
g
32
6
65
41
19
−9
−1
69
74
21
6
6
2
98
33
−5
3
3
9
−15
0