Lecture 14

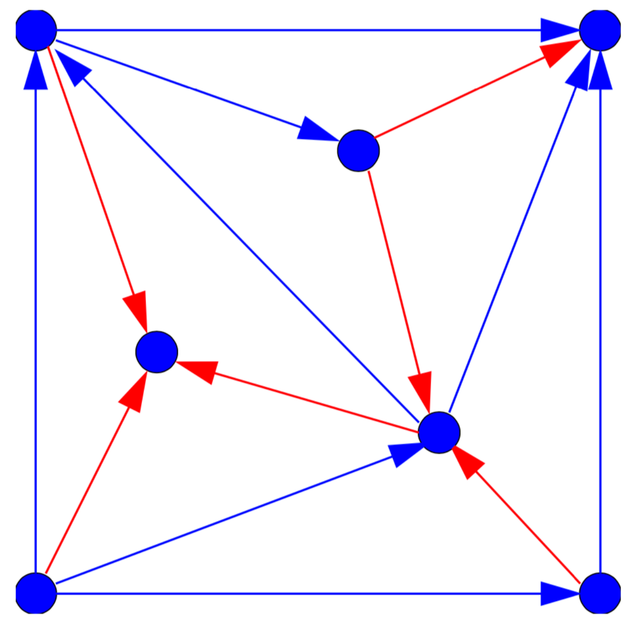
Last time: Network Flow Problems (V14 & GDs notes)

- Introduction
- Tree solutions

Today: Network Flow Problems (V14 & GDs notes)

- Tree solutions correspond to basic solutions
- Primal and dual network simplex method
- Integrality

Recap - spanning trees



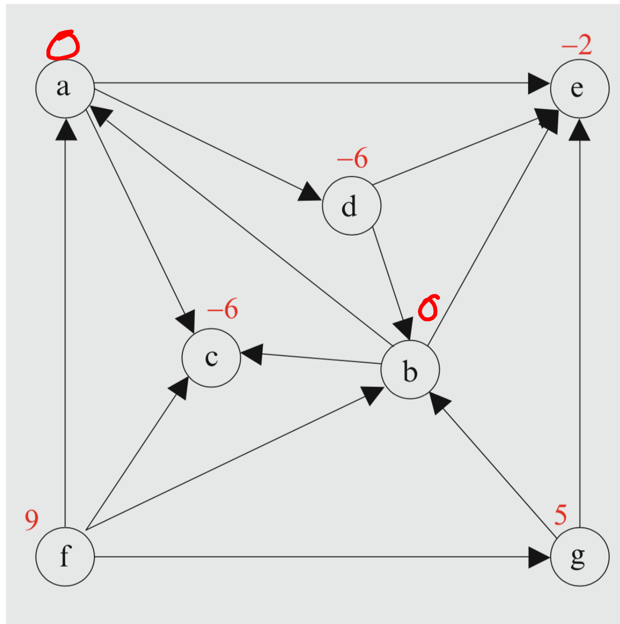
Spanning Tree – A tree touching every node

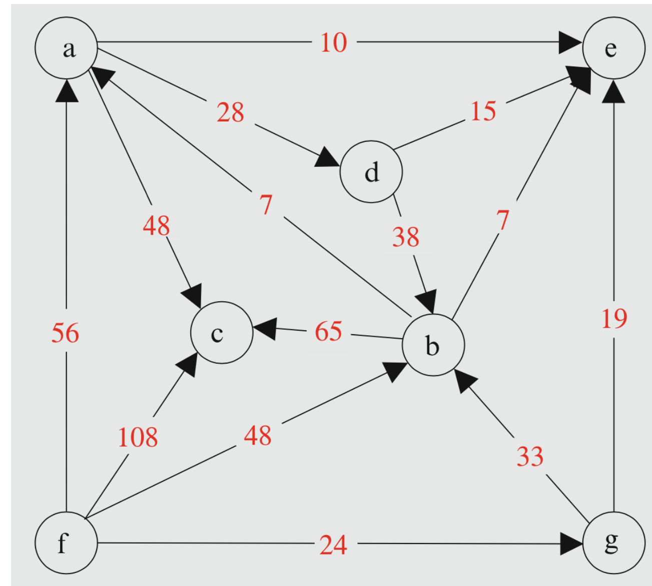
Remove edge – the tree is disconnected

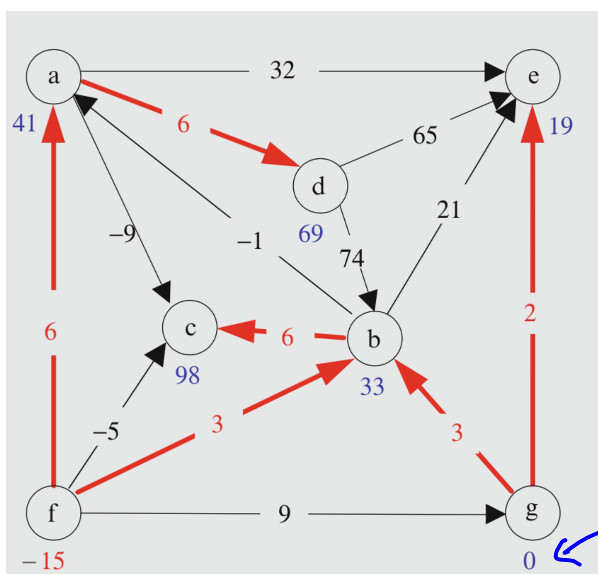
Add edge – make a cycle

Exactly $n-1$ edges where $n = |V|$

Recap - Tree solution







x - primal flows (on tree edges)

y - dual variables

z - dual slacks (on other edges)

$$y_c - y_a + z_{ac} = c_{ac}$$

root node

Redundancy

The columns sum to zero

$$x^T = [\; x_{\rm ac} \; x_{\rm ad} \; x_{\rm ae} \; x_{\rm ba} \; x_{\rm bc} \; x_{\rm be} \; x_{\rm db} \; x_{\rm de} \; x_{\rm fa} \; x_{\rm fb} \; x_{\rm fc} \; x_{\rm fg} \; x_{\rm gb} \; x_{\rm ge} \;],$$

$$A = \begin{bmatrix} -1 & -1 & -1 & 1 & & & & & 1 & & & & & \\ & & & -1 & -1 & -1 & 1 & & & 1 & & & 1 & \\ 1 & & & & 1 & & & & & & 1 & & & \\ & 1 & & & & & -1 & -1 & & & & & & \\ & & \cancel{1} & & & \cancel{1} & & \cancel{1} & & & & & & \cancel{1} \\ & & & & & & & & -1 & -1 & -1 & -1 & & \\ & & & & & & & & & & & 1 & -1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ -6 \\ -6 \\ \cancel{-2} \\ 9 \\ 5 \end{bmatrix}$$

$$c^T = [\; 48 \; 28 \; 10 \; 7 \; 65 \; 7 \; 38 \; 15 \; 56 \; 48 \; 108 \; 24 \; 33 \; 19 \;].$$

There is one redundant equation

Reduced system with one (arbitrary) equation removed:

$$\tilde{A}x = -\tilde{b} \qquad (m-1)\times n$$

Equivalent LP:

$$\min \; c^T x$$
$$\text{s.t. } \tilde{A}x = -\tilde{b}$$
$$x \geq 0$$

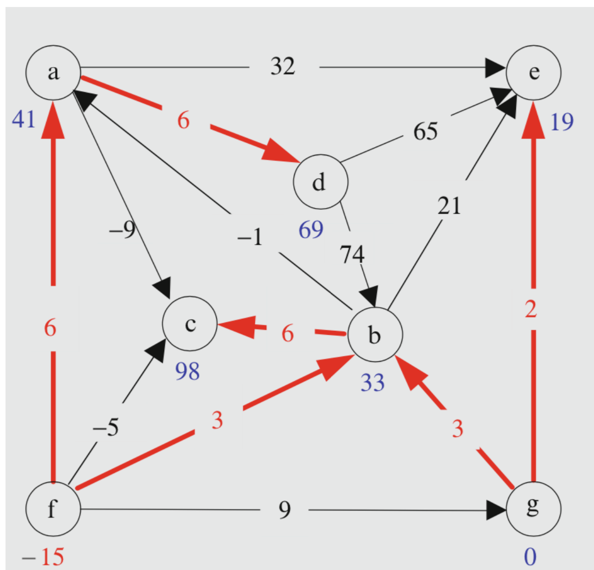
where $\tilde{A}$ has full row-rank

Recall: A basis in $\tilde{A}$ corresponds to a nonsingular $(m-1)\times(m-1)$ submatrix

Spanning trees correspond to basic solutions

THEOREM 14.1. *A square submatrix of $\tilde{A}$ is a basis if and only if the arcs to which its columns correspond form a spanning tree.*

„Proof": Rearrange rows and columns according to „computation order"



$$
\begin{array}{c}
\\ d \\ a \\ f \\ c \\ b \\ e
\end{array}
\begin{array}{c}
\begin{array}{cccccc} (a,d) & (f,a) & (f,b) & (b,c) & (g,b) & (g,e) \end{array} \\
\begin{bmatrix}
1 & & & & & \\
-1 & 1 & & & & \\
 & -1 & -1 & & & \\
 & & & 1 & & \\
 & & 1 & -1 & 1 & \\
 & & & & & 1
\end{bmatrix}
\end{array}
$$

Nonsingular: lower triang.

The columns in the submatrix corresponds to basic variables

$$
x^T = \begin{bmatrix} x_{ac} & x_{ad} & x_{ae} & x_{ba} & x_{bc} & x_{be} & x_{db} & x_{de} & x_{fa} & x_{fb} & x_{fc} & x_{fg} & x_{gb} & x_{ge} \end{bmatrix},
$$

$$
A = \begin{bmatrix}
-1 & -1 & -1 & 1 & & & & & 1 & & & & & \\
 & & & -1 & -1 & -1 & 1 & & & 1 & & & 1 & \\
1 & & & & 1 & & & & & & 1 & & & \\
 & 1 & & & & & -1 & -1 & & & & & & \\
 & & 1 & & & 1 & & 1 & & & & & & 1 \\
 & & & & & & & & -1 & -1 & -1 & -1 & & \\
 & & & & & & & & & & & 1 & -1 & -1
\end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ -6 \\ -6 \\ -2 \\ 9 \\ 5 \end{bmatrix}
$$

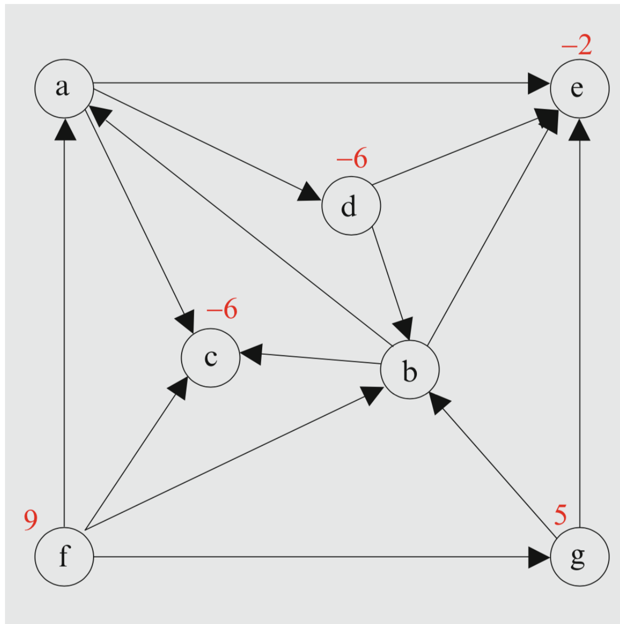
$$
c^T = \begin{bmatrix} 48 & 28 & 10 & 7 & 65 & 7 & 38 & 15 & 56 & 48 & 108 & 24 & 33 & 19 \end{bmatrix}.
$$

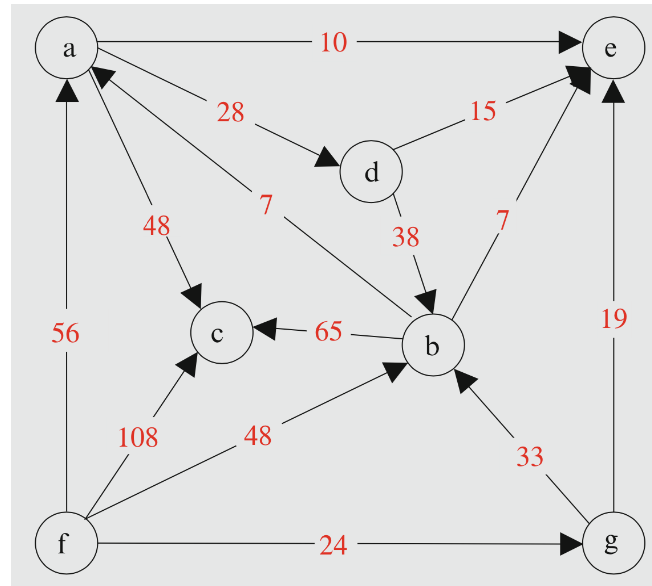
Network Simplex Algorithms

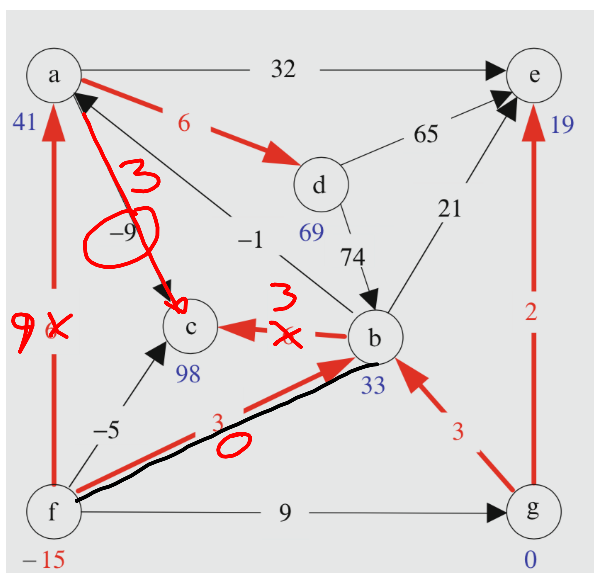
May as well rely on spanning tree solutions for our simplex methods.

Next: specialized simplex methods (see the book/lecture notes for more)

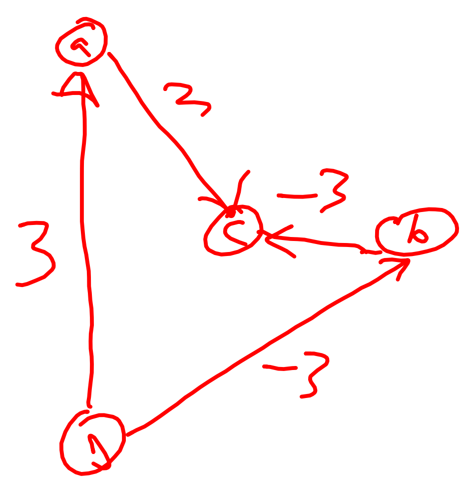
First pivot - primal flow







Max added cycle-flow without breaking constraints:

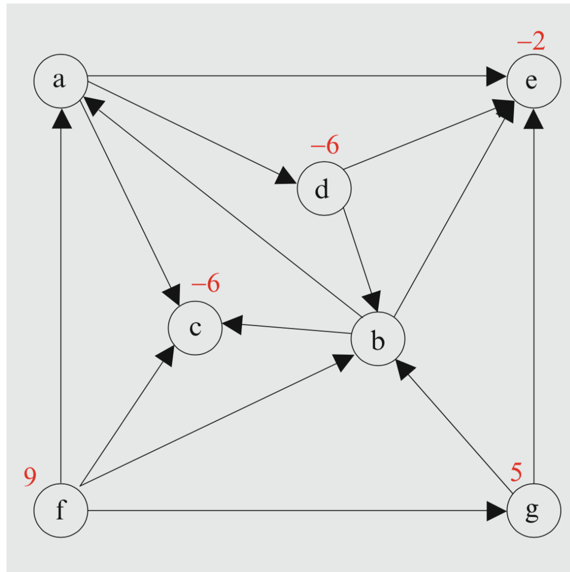


The edge (a,c) is entering (positive flow)

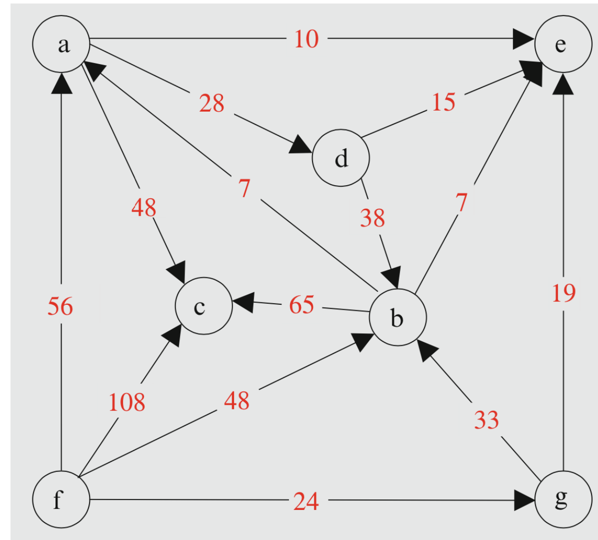
The edge (f,b) is leaving (zero flow)

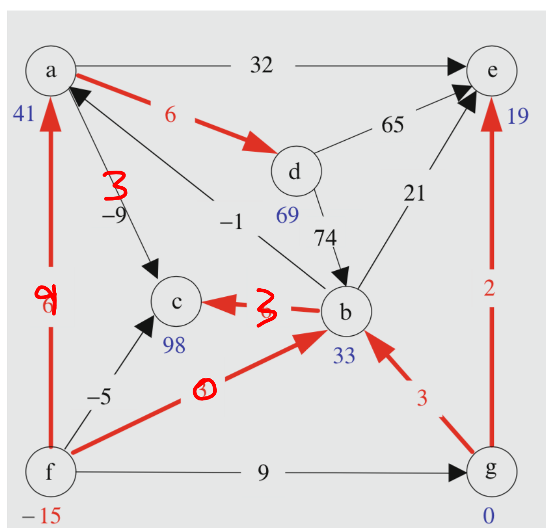
First pivot - dual variables

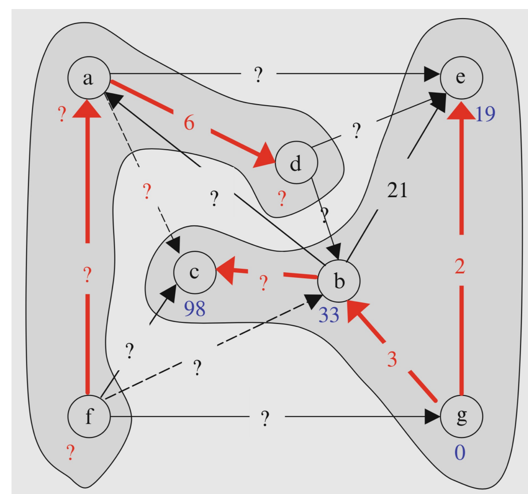
Supply/demand



Costs





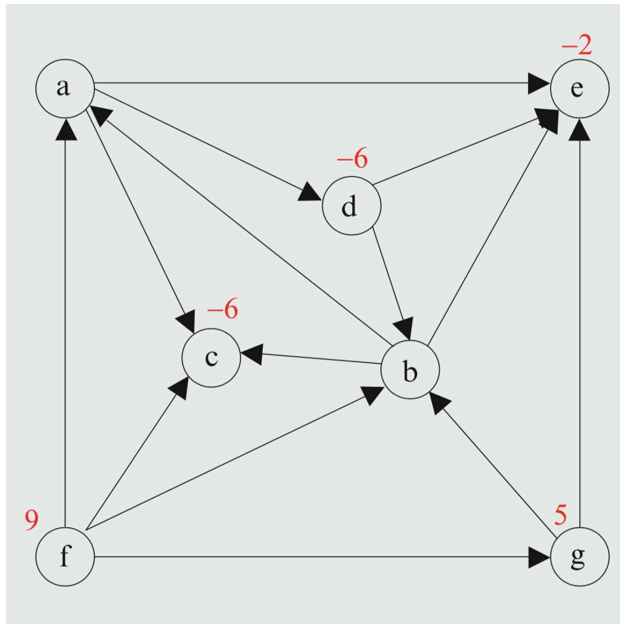


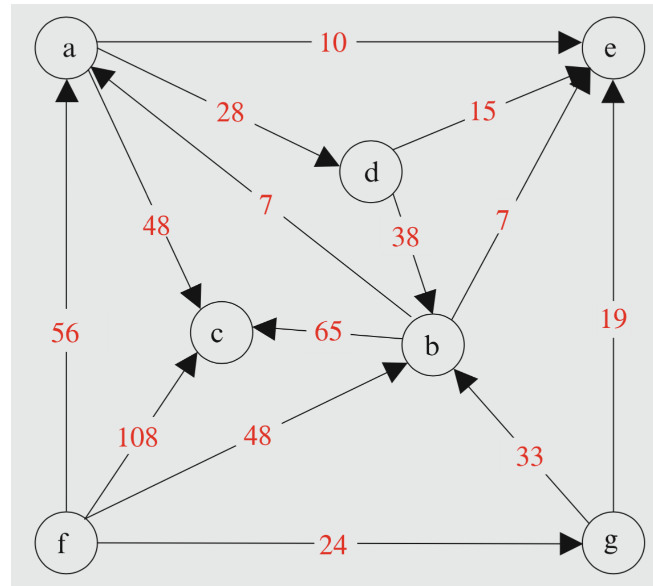
Dual constraint $\quad y_c - y_a + z_{ac} = c_{ac}$

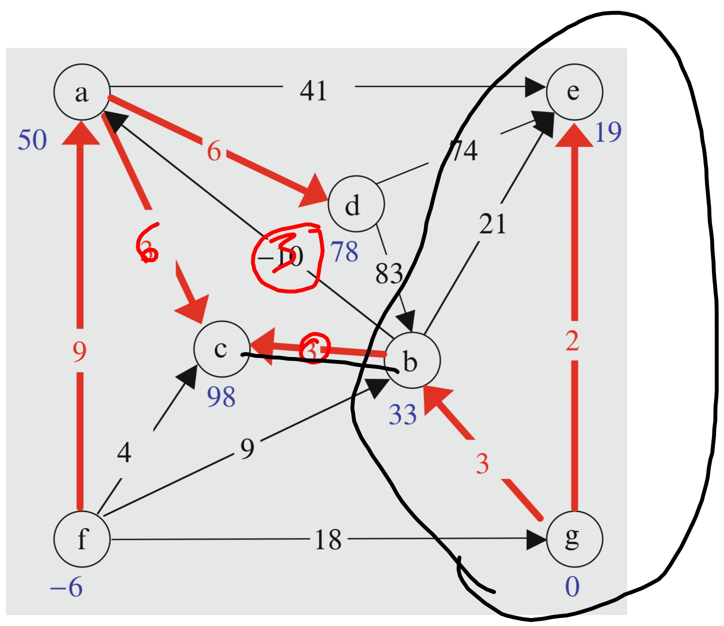
$\qquad\qquad\qquad\qquad +9 \qquad +9$

Exlanation: since the dual slack increase from -9 to zero, the dual variable must increase by the same amount to satisfy the dual constraint on (b,a)

First pivot - result



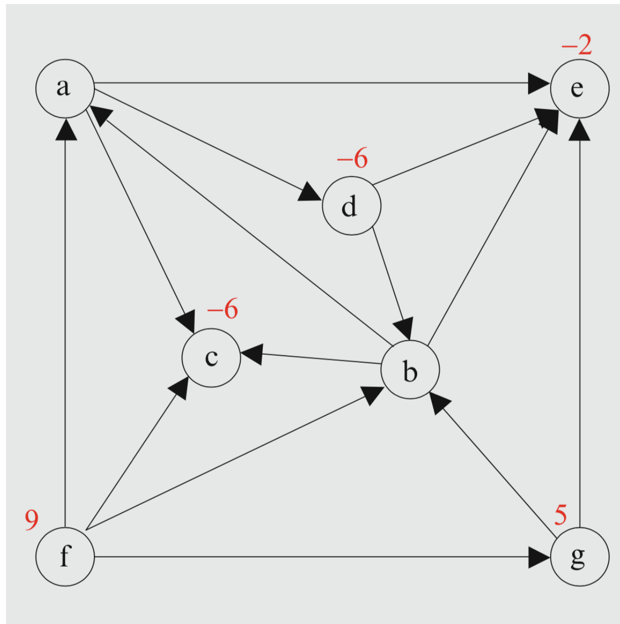


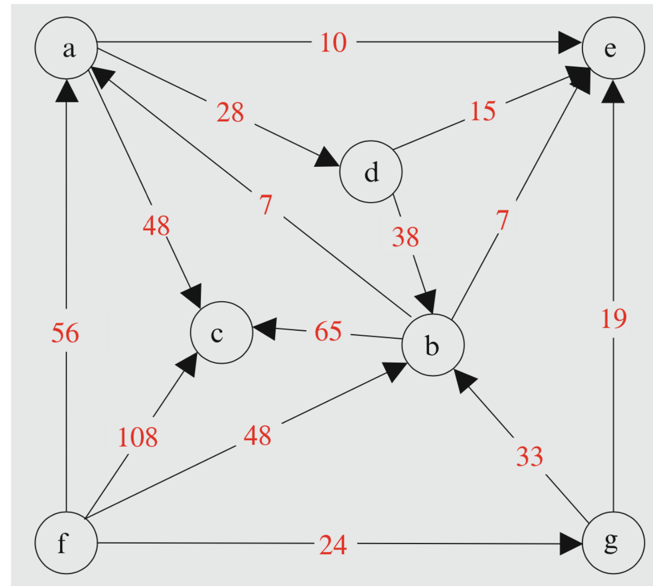


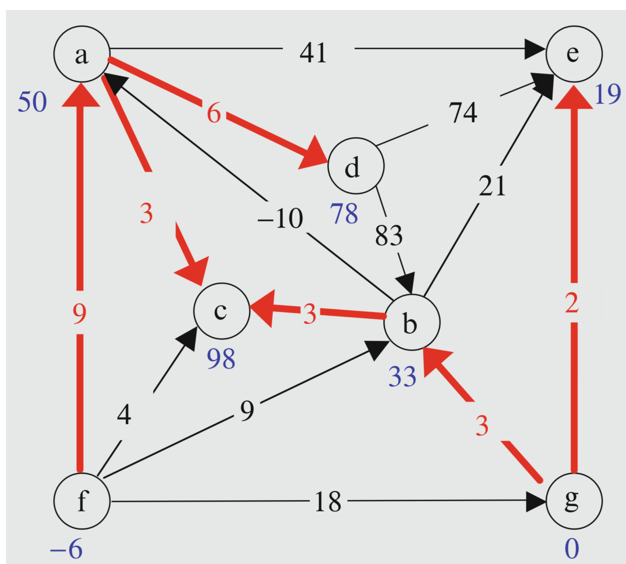
$Y_a - Y_b + Z_{ba} = C_{ba}$

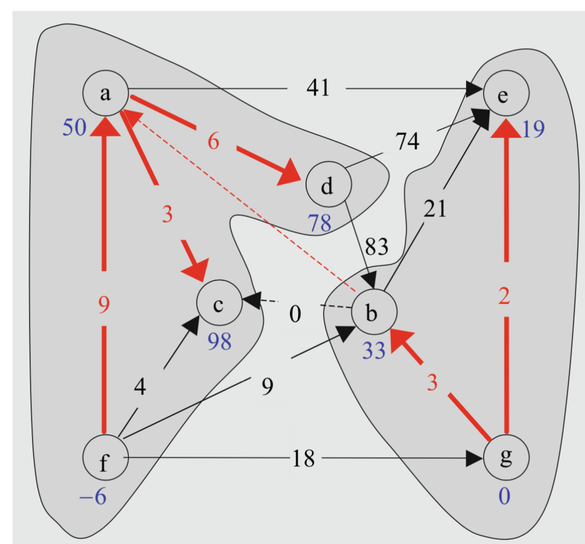
-10 $+10$

Second pivot

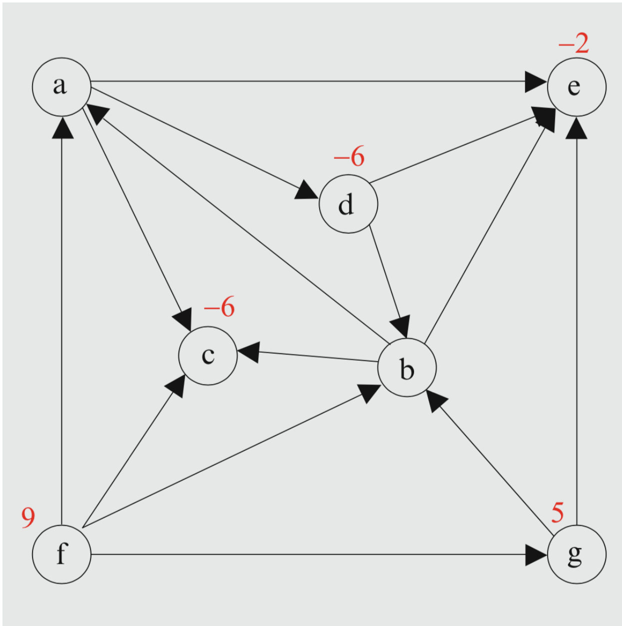




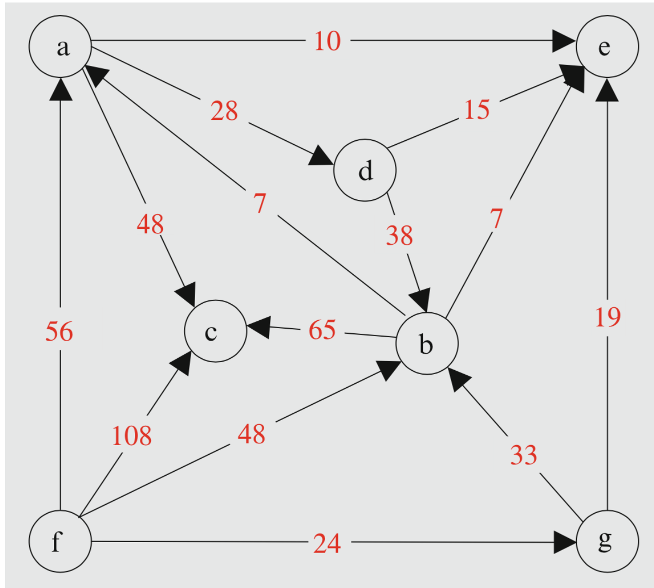




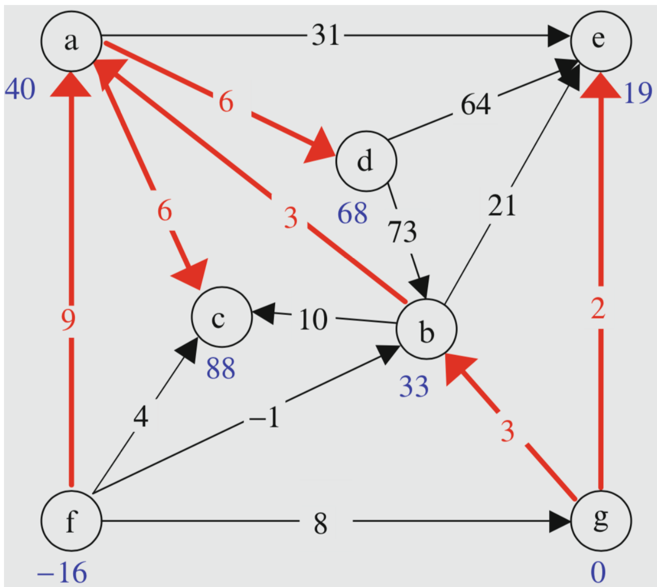
Third pivot



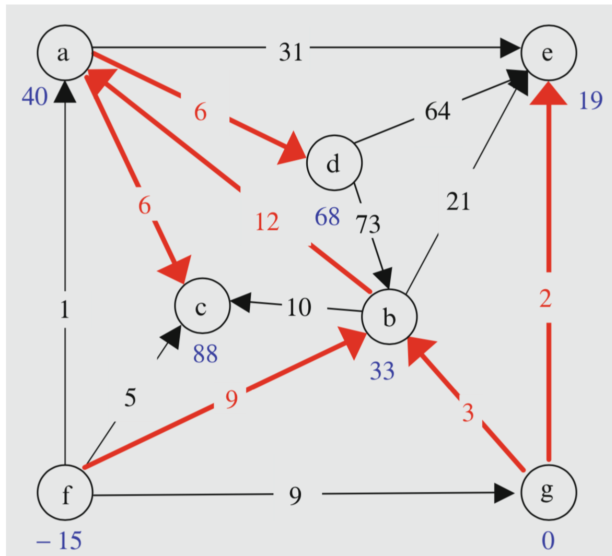
a
e
d
c
b
f
g
−2
−6
−6
9
5



a
e
d
c
b
f
g
10
28
15
48
7
38
7
56
65
19
108
48
33
24



a
e
d
c
b
f
g
31
6
64
40
19
6
3
68
73
21
9
10
2
88
33
4
−1
3
8
−16
0



a
e
d
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b
f
g
31
6
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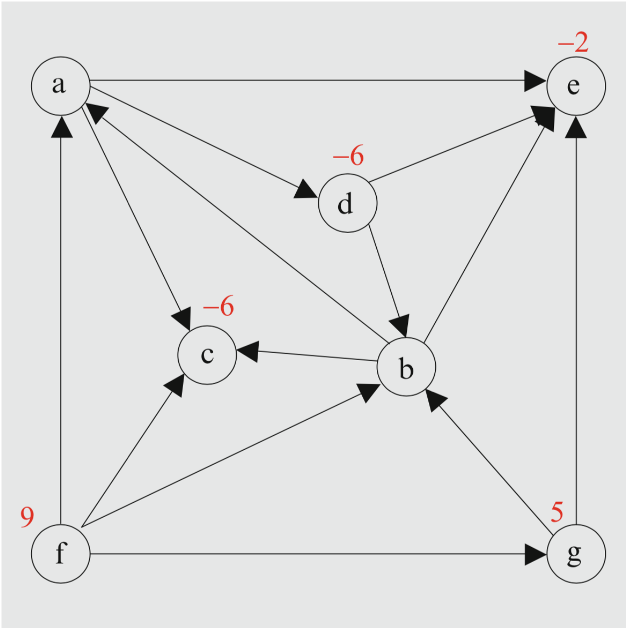
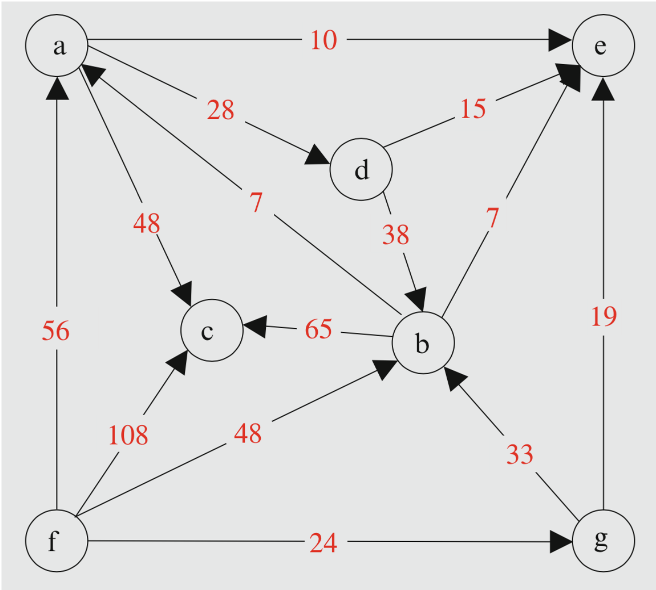
40
19
6
12
68
73
21
1
10
2
88
33
5
9
3
9
− 15
0

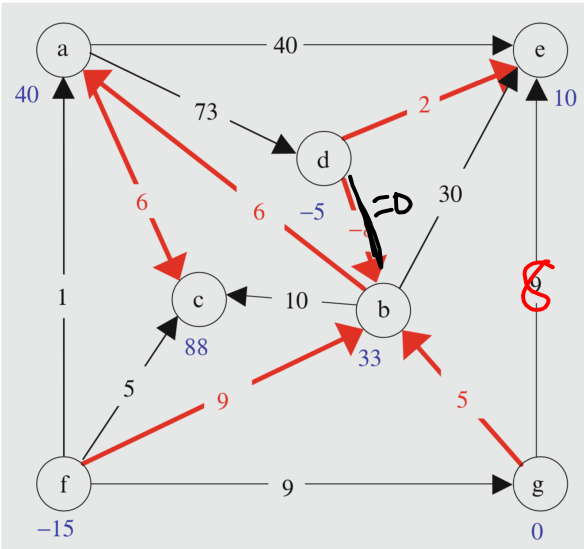
Some comments

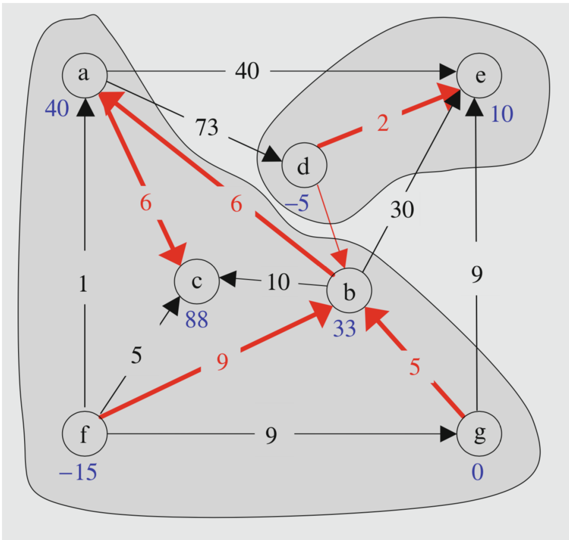
1. There is no need to recompute y in every iteration
2. Updating the dual slacks is the most costly for big problems
3. Degenerate solutions are common but not problematic, can be avoided
4. If a cycle has no "backward edges" the problem may be unbounded
5. The method can easily be adapted to accommodate capacity constraints on edges, i.e. $x_{uv} \le a_{uv}$

Dual network simplex method

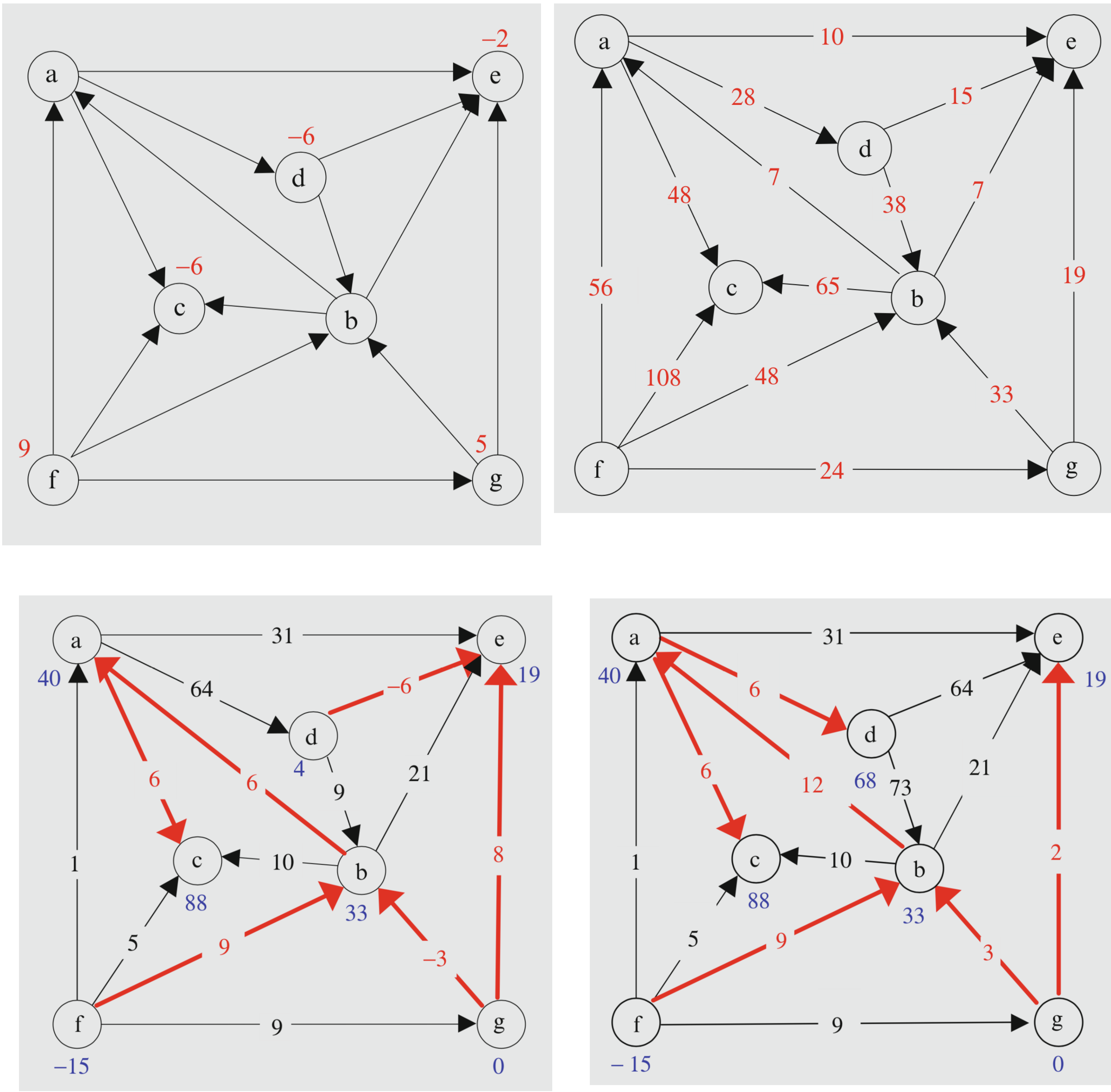








First dual pivot - result



How to find initial feasible solution?

Two-step method like we have seen before, adapted to networks:

1. Use modified costs $c=0$. Then the dual problem is feasible, and the dual simplex method can be used to get a feasible solution
2. Switch to the original cost c and use the simplex method