Lecture 16 - last lecture

Last time

- Integrality (V14)
- Applications (V14-15)
  - Transportation
  - Matching/assignment
  - Shortest paths

Today

- Summary of curriculum
- Exam problems from 2017

The rest of the semester

- Tutorial tomorrow and next wednesday
- Guest lecture next tuesday
- Q&A june 5.
- Exam june 7.

The **final** syllabus is the union of

(i) those topics that are lectured (see lecture notes)

(ii) The lecture notes of G. Dahl, including "A mini-introduction to convexity"

(iii) the following from Vanderbei's book (see below):

- Chapter 1-6: all.
- Chapter 7: 7.1.
- Chapter 11: 11.1-11.3.
- Chapter 12: 12.4.
- Chapter 14: all sections except 14.5.
- Chapter 15: 15.1-15.3.
- Chapter 17: all.

(Here, for instance, 11.1-11.3 means, 11.1, 11.2 and 11.3.)

We use the following edition of Vanderbeis book:

- R. Vanderbei, "*Linear Programming: Foundations and Extensions*". Fourth Edition, Springer (2014). It may be downloaded for free, see More

Problem 1

Consider the LP problem (P)

$$\begin{array}{rrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & -x_1 + 3x_2 & \leq 3, \\ & 2x_1 + x_2 & \leq 8, \\ & x_1, x_2 & \geq 0. \end{array}$$

1a

Use the simplex method to find an optimal solution and the corresponding objective value.

$$\eta = x_1 + 2x_2$$

$$\rightarrow w_1 = 3 + x_1 - 3x_2$$

$$w_2 = 8 - 2x_1 - x_2 \qquad w_1, w_2 \geq 0$$

$$\eta = x_1 + 2\left(1 + \tfrac{1}{3}x_1 - \tfrac{1}{3}w_1\right) = 2 + \tfrac{5}{3}x_1 - \tfrac{2}{3}w_1$$

$$x_2 = 1 + \tfrac{1}{3}x_1 - \tfrac{1}{3}w_1$$

$$\rightarrow w_2 = 8 - 2x_1 - \left(1 + \tfrac{1}{3}x_1 - \tfrac{1}{3}w_1\right) = 7 - \tfrac{7}{3}x_1 + \tfrac{1}{3}w_1$$

$$\eta = 2 + \tfrac{5}{3}\left(3 - \tfrac{3}{7}w_2 + \tfrac{1}{7}w_1\right) - \tfrac{2}{3}w_1 = 7 - \tfrac{5}{7}w_2 - \tfrac{3}{7}w_1$$

$$x_2 = 2 - \tfrac{1}{7}w_2 - \tfrac{2}{7}w_1$$

$$x_1 = 3 - \tfrac{3}{7}w_2 + \tfrac{1}{7}w_1$$

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optimal and feasible!

Basic solution: $w_1 = w_2 = 0$

$x_1 = 3$ and $x_2 = 2$

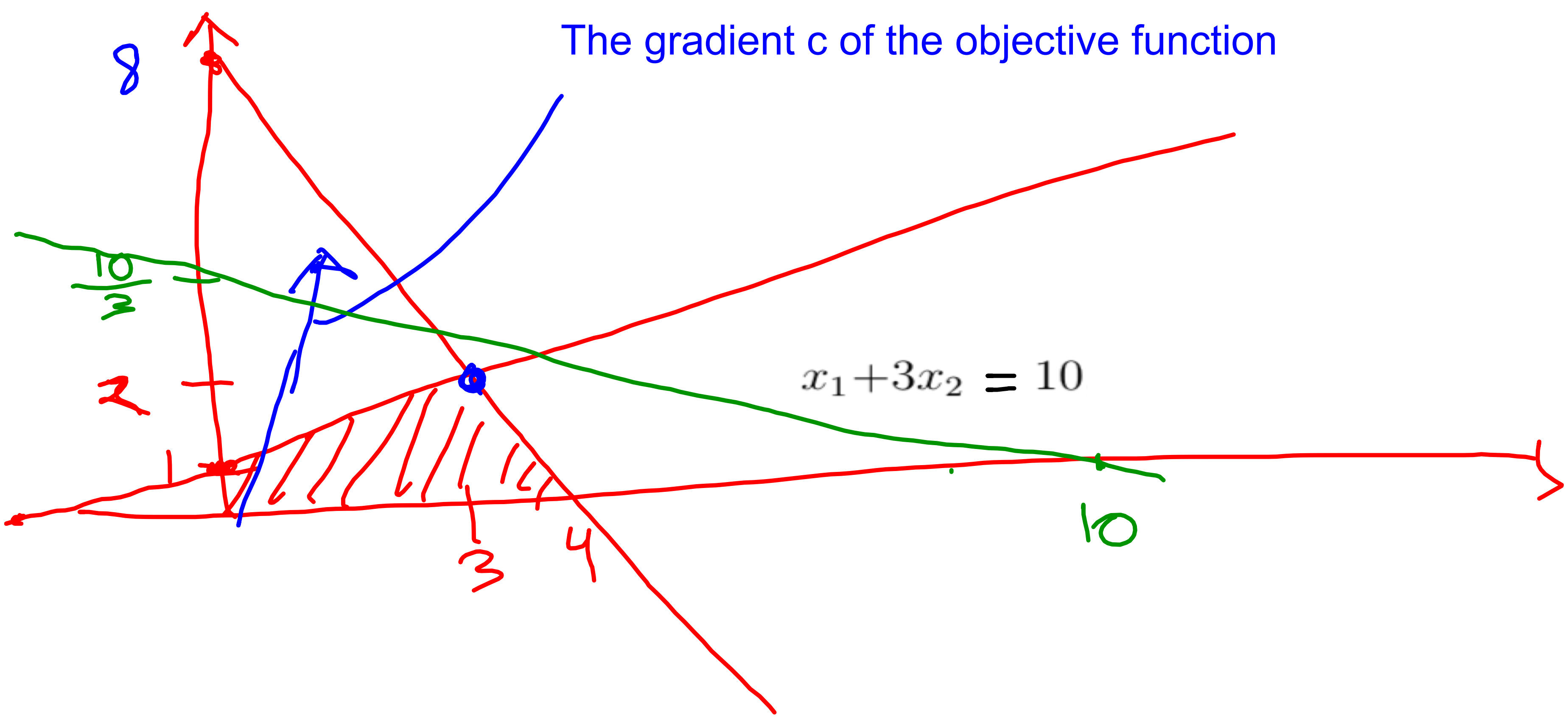
$\eta = 7$

$$
\begin{array}{lrl}
\text{maximize} & x_1 + 2x_2 & \\
\text{subject to} & -x_1 + 3x_2 & \leq 3, \\
 & 2x_1 + x_2 & \leq 8, \\
 & x_1, x_2 & \geq 0.
\end{array}
$$

$x_2 \leq 1 + \frac{1}{3} x_1$

1b

Make a plot of the feasible region for (P) and indicate your optimal solution.



$$\begin{array}{lrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & -x_1 + 3x_2 & \leq 3, \\ & 2x_1 + x_2 & \leq 8, \\ & x_1, x_2 & \geq 0. \end{array}$$

1c

(i) Let (P′) be the LP problem formed by adding the constraint, $x_1 + 3x_2 \leq 10$, to (P). What is the optimal objective value of (P′)?
(ii) Let (P″) be the LP problem formed by replacing the objective function of (P) by $x_1 + 3x_2$. What is the optimal objective value of (P″)?

Test the previous optimal dictionary:

$$\eta = x_1 + 3x_2 = 3 - \frac{3}{7} w_2 + \frac{1}{7} w_1 + 3\left(2 - \frac{1}{7} w_2 - \frac{2}{7} w_1\right)$$

$$= 9 - \frac{6}{7} w_2 - \frac{5}{7} w_1$$

Optimal solution (negative coefficients)!

Optimal objective value is 9

$$\begin{array}{lrl} \text{maximize} & x_1 + 2x_2 & \\ \text{subject to} & -x_1 + 3x_2 & \leq 3, \\ & 2x_1 + x_2 & \leq 8, \\ & x_1, x_2 & \geq 0. \end{array}$$

1d

What is the dual problem (D) of (P)?

$$\begin{array}{ll} \min & 3y_1 + 8y_2 \\ \text{s.t.} & -y_1 + 2y_2 \geq 1 \\ & 3y_1 + y_2 \geq 2 \\ & y_1, y_2 \geq 0 \end{array}$$

1e

What is an optimal solution to (D) and what is the corresponding optimal value? use optimal primal dictionary

$$\begin{array}{rcrcrcr} \eta & = & 7 & - & (5/7)w_2 & - & (3/7)w_1 \\ \hline x_2 & = & 2 & - & (1/7)w_2 & - & (2/7)w_1 \\ x_1 & = & 3 & - & (3/7)w_2 & + & (1/7)w_1 \end{array}$$

Dual optimal dictionary by negative transpose:

$$-\xi = 7 - 2z_2 - 3z_1$$

$$y_2 = \frac{5}{7} + \frac{1}{7}z_2 + \frac{3}{7}z_1$$

$$y_1 = \frac{3}{7} + \frac{2}{7}z_2 - \frac{1}{7}z_1$$

optimal solution $(y_1, y_2) = \left(\frac{3}{7}, \frac{5}{7}\right)$

Optimal value = 7

Problem 2

A matrix game is determined by a matrix $A = [a_{ij}]_{i=1,\dots,m,j=1,\dots,n}$. The row player (R) pays a_{ij} kroner to the column player (K) if R chooses option i and K chooses option j. R is playing with a randomized strategy $y = (y_1, \dots, y_m)^T$, choosing option i with probability y_i, where $y_i \geq 0$ and $\sum_{i=1}^m y_i = 1$. Similarly, K is playing with a randomized strategy $x = (x_1, \dots, x_n)^T$, where $x_j \geq 0$ and $\sum_{j=1}^n x_j = 1$.

2a

(i) If K uses a fixed strategy x, what is R's corresponding best defence, i.e., best corresponding strategy y?
(ii) If R adopts the strategy in (i) in defence of the strategy x chosen by K, what is K's best strategy x^*?

Expected payoff $\quad y^T A x$

i) Row-player should minimize this

$$\min_y \; y^T A x \quad \text{s.t } e^T y = 1,\ y \geq 0$$

ii) Column-player should maximize payoff

$$\max_x \; \min_y \; y^T A x \quad \text{s.t } e^T x = 1,\ x \geq 0$$

2b

Explain how part (ii) of the last problem can be formulated as the LP problem

$$\begin{array}{ll} \text{maximize} & v \\ \text{subject to} & v \leq e_i^T A x, \quad i = 1, 2, \ldots, m, \\ & \sum_{j=1}^{n} x_j = 1, \\ & x_j \geq 0, \qquad j = 1, 2, \ldots, n, \end{array}$$

where $e_i \in \mathbb{R}^m$ is the vector of all zeros with 1 in the i-th position.

„Inner" optimization

$$\min_y \; y^T \underbrace{Ax}_{\text{given}} \quad \text{s.t.} \quad \underbrace{e^T y = 1, \; y \geq 0}_{\text{polytope, convex set with vertices } \{e_i\}}$$

$$= \min_i e_i^T A x$$

So

$$\max_x \min_y y^T A x = \max_x \min_i e_i^T A x$$

Let $v \in \mathbb{R}$ be a lower bound on $e_i^T A x$

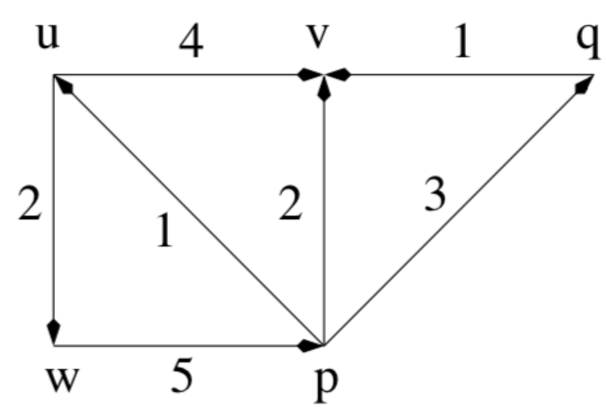
i.e. $v \leq \min_i e_i^T A x$

Maximizing v yields the LP-problem above

Problem 3

Consider the minimum cost network flow problem based on the directed graph shown in the figure. The number associated with each directed edge (i, j) is its cost $c_{i,j}$ (per unit flow). At each node i let b_i be its supply. The supplies are

$$b_u = 1, \quad b_v = -2, \quad b_w = -3, \quad b_p = 6, \quad b_q = -2.$$



3a

Write down the flow balance equation at node i. Let T_1 be the spanning tree consisting of the edges

$$(u, w), \quad (p, u), \quad (p, q), \quad (q, v),$$

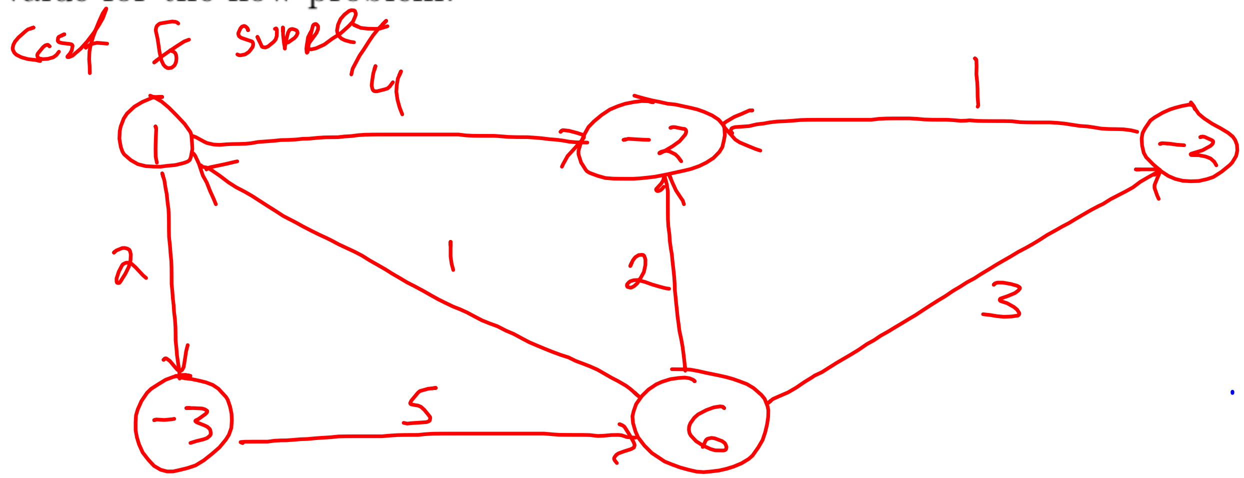
and all their nodes. Compute the tree solution x corresponding to T_1.

Flow balance at node i

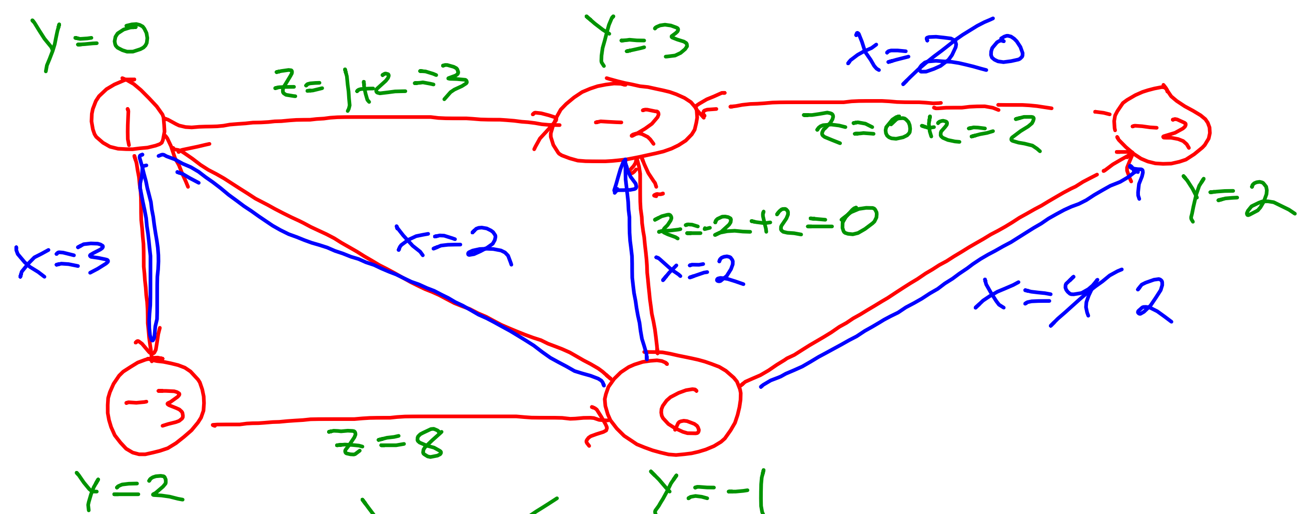
$$\sum_{j:(j,i)\in E} x_{ji} - \sum_{j:(i,j)\in E} x_{ij} = -b_i$$

3b

Use the network simplex method to find an optimal solution and optimal value for the flow problem.



Flow:



$$y_j - y_i + \cancel{z_{ij}} = c_{ij} \quad \text{for tree edges}$$

$$z_{ij} = 0$$

$$z_{ij} = c_{ij} - (y_j - y_i) \quad \text{for non-tree edges}$$

3c

In a general network flow problem, there may or may not be a unique optimal solution.
(i) Suppose the optimal solution is unique. If the supplies b_i are integers, will the flows x_{ij} in the optimal solution be integers?
(ii) Suppose there is more than one optimal solution. If the supplies b_i are integers, will the flows x_{ij} in an optimal solution be integers?
Explain your answers.

i) Yes. Integer data and no divisions in the algorithm

ii) Not necessarily. Any convex combination of solutions is also a solution, and will not be integral in general