$\underline{P}$ $\max\ p \cdot x$
s.t. $q \cdot x \leq \beta$
$x_j \leq 1 \quad j=1,\dots,m$
$x_j \geq 0 \quad j=1,\dots,m$

$\underline{D}$ $\min\ \beta y_0 + \sum_{j=1}^{m} y_j$
s.t. $q_j y_0 + y_j \geq p_j \quad \forall j=1,\dots,m$
$y_j \geq 0 \quad \forall j=1,\dots,m$

$O\underline{P}.\ p \cdot \bar{x} = \sum_{j=1}^{m} p_j \bar{x}_j$

$p_K \bar{x}_K + \sum_{j>K} 1 \cdot p_j$

$= \frac{p_K}{q_K}\left(\beta - \underbrace{q_{K+1} - \dots - q_m}\right) + \sum_{j>K} p_j$

$O\ \underline{D}\ \beta \bar{y}_0 + \sum_{j=1}^{m} \bar{y}_j$

$= \beta \frac{p_K}{q_K} + \sum_{j>K} q_j\left(\frac{p_j}{q_j} - \frac{p_K}{q_K}\right)$

$\underline{OK}$

need to show $\bar{x}$ and $\bar{y}$ are feasible

$\underline{\bar{P}}$ $\sum_{j=1}^{m} q_j \bar{x}_j \overset{?}{\leq} \beta$

$\sum_{j=1}^{n} q_j \bar{x}_j = q_K \bar{x}_K + \sum_{j>K} q_j$

$= \left(\beta - \sum_{j>K} q_j\right) + \sum_{j>K} q_j = \beta$

$\underline{ok}$

if $\neq K \quad 0 \leq x_j \leq 1$

$K \ \min\{j : \sum_{i>j} q_i \leq \beta\}$

$\Rightarrow \sum_{i>K} q_i \leq \beta$

$\Rightarrow \beta - \sum_{i>K} q_i \geq 0$

$\Rightarrow \bar{x}_K \geq 0$

$\bar{x}_K \leq 1$ (assumption)

$\Rightarrow$ $\underline{\bar{x} \text{ is } P\text{-feasible}}$

$\underline{\bar{D}}$ $1 \leq j \leq K$

$\Rightarrow \bar{y}_j = 0$

$q_j \bar{y}_0 + \bar{y}_j = q_j \frac{p_K}{q_K} + 0 \geq p_j$

since $\frac{p_j}{q_j} \leq \frac{p_K}{q_K}$ ok

$o\ j>K \quad \bar{y}_j = q_j\left(\frac{p_j}{q_j} - \frac{p_K}{q_K}\right)$

$q_j \bar{y}_0 + \bar{y}_j = \cancel{q_j \frac{p_K}{q_K}} + q_j \frac{p_j}{q_j} - \cancel{q_j \frac{p_K}{q_K}}$

$= p_j \geq p_j$

$\bar{y}_j \geq 0 \quad \forall j=1,\dots,m$?

if $j=1,\dots,K \Rightarrow \bar{y}_j = 0 \Rightarrow$ ok

if $j>K \quad \frac{p_j}{q_j} > \frac{p_K}{q_K} \Rightarrow q_j\left(\frac{p_j}{q_j} - \frac{p_K}{q_K}\right) > 0$

$\Rightarrow p_j > \frac{p_K q_j}{q_K}$

$\Rightarrow p_j - \frac{p_K q_j}{q_K} > 0$

$\Rightarrow q_j\left(\frac{p_j}{q_j} - \frac{p_K}{q_K}\right) > 0$

$\Rightarrow \bar{y}_j > 0$

if $j=0 \Rightarrow \bar{y}_0 = \frac{p_K}{q_K} > 0$