1 $B(0,1)$

$S=$ 

$x, y \in B(0,1)$, $\|x\| \leq 1$, $\|y\| \leq 1$

$\lambda \in \mathbb{R}$, $0 \leq \lambda \leq 1$

$z := \lambda x + (1-\lambda) y$

$\|z\| = \|\lambda x + (1-\lambda) y\| \leq \|\lambda x\| + \|(1-\lambda) y\|$ (by $\triangle$ ineq.)

$= |\lambda| \cdot \|x\| + |1-\lambda| \|y\|$

$= \lambda \|x\| + (1-\lambda) \|y\|$

$\leq \lambda + (1-\lambda) = 1$

$\Rightarrow z \in B(0,1)$

$\Rightarrow B(0,1)$ is convex

$B(a,r)$ is obtained from $B(0,1)$ via scaling and translation

$\Rightarrow B(a,r)$ is convex

2 $L \triangleleft \mathbb{R}^n$ $(x, y \in L,\ \alpha \in \mathbb{R} \Rightarrow x+y \in L,\ \alpha x \in L)$

$x, y \in L$, $\lambda \in \mathbb{R}$ s.t. $0 \leq \lambda \leq 1$

claim $\lambda x + (1-\lambda) y \in L$

$\lambda x \in L$, $(1-\lambda) y \in L$

$\Rightarrow \lambda x + (1-\lambda) y \in L$

$\Rightarrow L$ is convex

3 NO A . . . . B

$S = \{A, B\}$ is not convex

4 Ω is a family of convex sets $\subseteq \mathbb{R}^n$

$I := \{x \in \mathbb{R}^n \text{ s.t. } x \in C \quad \forall C \in \Omega\}$

$= \bigcap_{C \in \Omega} C$

$x, y \in I$, $0 \leq \lambda \leq 1$

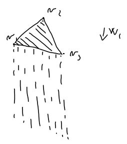
we claim that $\lambda x + (1-\lambda) y \in I$

$\Leftrightarrow \lambda x + (1-\lambda) y \in C \quad \forall C \in \Omega$

Choose $C \in \Omega$. $x, y \in I \Rightarrow x, y \in C$

C convex $\Rightarrow \lambda x + (1-\lambda) y \in C$

$\Rightarrow I$ is convex



5 Suppose B is a polyhedron.

$B \neq \emptyset \Rightarrow$ apply THM

$\Rightarrow \exists v_1, \ldots, v_n$, $w_1, \ldots, w_t \in \mathbb{R}^n$ s.t.

$B = \left\{ \sum_{i=1}^{n} \lambda_i v_i + \sum_{j=1}^{t} \mu_j w_j \text{ s.t. } \lambda_i, \mu_j \geq 0, \ \sum_{i=1}^{n} \lambda_i = 1 \right\}$

B is bounded $\Rightarrow t = 0$

$B = \left\{ \sum_{i=1}^{n} \lambda_i v_i \text{ s.t. } \lambda_i \geq 0, \ \sum \lambda_i = 1 \right\}$

we can choose $v_1, \ldots, v_n$ to be the extreme points of B. $\Rightarrow \#\{\text{extreme points of } B\} < \infty$

claim $\forall x \in B$ s.t. $\|x\| = 1$, x is extreme.

PROOF

Suppose $x = \frac{1}{2} x_2 + \frac{1}{2} x_2$,

$x_1, x_2 \in B$

$\|x_1\| \leq 1$, $\|x_2\| \leq 1$

$\Rightarrow 1 = \|x\| = \|\frac{1}{2} x_1 + \frac{1}{2} x_2\| \leq \|\frac{x_1}{2}\| + \|\frac{x_2}{2}\|$

$= \frac{\|x_1\|}{2} + \frac{\|x_2\|}{2} \leq \frac{1}{2} + \frac{1}{2} = 1$

$\Rightarrow$ all the inequalities are equalities

$\Rightarrow \|x_1\| = 1$, $\|x_2\| = 1$,

$\|\frac{1}{2} x_1 + \frac{1}{2} x_2\| = \|\frac{1}{2} x_1\| + \|\frac{1}{2} x_2\|$

$\Rightarrow \|x_1 + x_2\| = \|x_1\| + \|x_2\|$

$\Rightarrow x_1$ and x_2 are l.d.

$\Rightarrow x_1 = \mu \cdot x_2 \Rightarrow$

$1 = \|x_1\| = \|\mu x_2\| = |\mu| \, \|x_2\| = |\mu| \Rightarrow \mu = \pm 1$