○ if $\mu = -1$, $x = x_1 + x_2 = x_1 - x_1 = 0$

$\Rightarrow \|x\| = 0 \neq 1 \Rightarrow \perp$

$\Rightarrow \mu = 1 \Rightarrow x_1 = x_2$

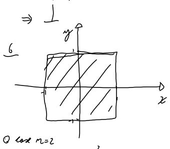
$\Rightarrow x = \frac{1}{2} x_1 + \frac{1}{2} x_2 = \frac{1}{2} x_1 + \frac{1}{2} x_1 = x_1 = x_2$

$\Rightarrow$ <u>x is an extreme point.</u>

$\Rightarrow$ {surface points of B} $\subset$ {extreme points of B}

$\Rightarrow \perp$

<u>6</u>



○ case $n=2$

$B_\infty = \{ (x,y) \in \mathbb{R}^2 \text{ s.t. } \|(x,y)\|_\infty \leq 1 \}$

$= \{ (x,y) \in \mathbb{R}^2 \text{ s.t. } \max\{|x|, |y|\} \leq 1 \}$

○ in general

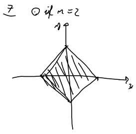
$B_\infty = \{ x \in \mathbb{R}^m : \|x\|_\infty \leq 1 \} = \{ x \in \mathbb{R}^m : \max_j |x_j| \leq 1 \}$

$= \{ x \in \mathbb{R}^m : |x_j| \leq 1 \quad \forall j = 1, .., m \}$

$= \{ x \in \mathbb{R}^m : -1 \leq x_j \leq 1 \quad \forall j = 1, .., m \}$

$\Rightarrow B_\infty$ is a polyhedron

<u>7</u> ○ if $n = 2$



$B = \{ (x,y) \in \mathbb{R}^2 \text{ s.t. } |x| + |y| \leq 1 \}$

○ <u>general case</u>

$\mathcal{E} := \{ (\varepsilon_1, ..., \varepsilon_m), \ \varepsilon_i \in \{-1, 1\}$

$\mathcal{E} = \{ (0, 0, ..., 0), (0, 0, ..., 0, 1), ... \}$

<u>claim</u> The equation $\sum_{j=1}^{m} |x_j| \leq 1$ (1)

is equivalent to the system of equations (2)

$$\sum_{j=1}^{m} \varepsilon_j x_j \leq 1 \quad \forall \varepsilon = (\varepsilon_1, ..., \varepsilon_m) \in \mathcal{E}$$

<u>PR.</u> x is a solution of (1) and choose $\varepsilon \in \mathcal{E}$.

$$\sum_{j=1}^{m} \varepsilon_j x_j \leq \left| \sum_{j=1}^{m} \varepsilon_j x_j \right| \leq \sum_{j=1}^{m} |\varepsilon_j x_j|$$

$$= \sum_{j=1}^{m} |\varepsilon_j| \cdot |x_j| = \sum_{j=1}^{m} |x_j| \leq 1$$

$\Rightarrow x$ is a solution of (2)

---

x is a sol of (2)

$\hat{\varepsilon} = (\text{sign}(x_1), \text{sign}(x_2), ..., \text{sign}(x_m)) \in \mathcal{E}$

$$\sum_{j=1}^{m} \hat{\varepsilon}_j x_j \leq 1$$

$$\Rightarrow \sum_{j=1}^{m} |x_j| \leq 1$$

$\Rightarrow x$ is a sol of (1)

$\Rightarrow B_1 = \{ x \in \mathbb{R}^m \text{ s.t. } \sum_{j=1}^{m} |x_j| \leq 1 \}$

$= \{ x \in \mathbb{R}^m \text{ s.t. } \sum_{j=1}^{m} \varepsilon_j x_j \leq 1 \ \forall \varepsilon \in \mathcal{E} \}$

$\Rightarrow B_1$ is a polyhedron.

<u>8</u> Set $x_i = x_i^+ - x_i^-$, $x_i^+, x_i^- \geq 0$

$\Rightarrow x = x^+ - x^-$, $x^+, x^- \geq 0$

$$\{ c^T x : Ax \leq b \} = \{ c^T (x^+ - x^-) : A(x^+ - x^-) \leq b, \ x^+, x^- \geq 0 \}$$

define $\tilde{c} = \begin{pmatrix} c \\ -c \end{pmatrix}$, $\tilde{A} = [A \ \ -A]$, $\tilde{x} = \begin{pmatrix} x^+ \\ x^- \end{pmatrix}$

$\tilde{c}^T \tilde{x} = c^T x^+ - c^T x^- = c^T x$,

$$\tilde{A} \tilde{x} = [A \ \ -A] \begin{bmatrix} x^+ \\ x^- \end{bmatrix} = A(x^+ - x^-) = A x$$

The problem can be rewritten as

$$\max \{ \tilde{c}^T \tilde{x} : \tilde{A} \tilde{x} \leq b, \tilde{x} \geq 0 \}$$

set $w := b - \tilde{A} \tilde{x}$, $c_1 = \begin{pmatrix} \tilde{c} \\ 0 \end{pmatrix}$, $x_1 = \begin{pmatrix} \tilde{x} \\ w \end{pmatrix}$

$A_1 = [\tilde{A} \ \ I]$

then $c_1^T x_1 = \tilde{c}^T \tilde{x}$, $A_1 x_1 = \tilde{A} \tilde{x} + w = b$

The problem becomes

$$\max \{ c_1^T x_1 : A_1 x_1 = b, x_1 \geq 0 \}$$