9$^{\text{th}}$ February, 2019

MAT3100 - SPRING 2019

Mandatory Assignment I

Submission deadline

Thursday 28$^{\text{th}}$ February 2019 (upload to Devilry).

Instructions

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you understand the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code (and a trial run) along with the rest of the assignment.

Please write down the solution to the exercises in English.

All mandatory assignments must be uploaded to Devilry.

- The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. If these two requirements are not met, the assignment will not be assessed.
- The submission must contain your name, course and assignment number.
- *To have a passing grade you must have satisfactory answers to at least 50% of the questions and have seriously attempted to solve all of them.*

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you must contact the Student Administration at the Dep. of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1

(a)

Let $b \in (0, \frac{1}{4})$ be a real number. Use the simplex method to find an optimal solution for the LP problem

$$\begin{array}{lrrrrrrcl} \text{maximize} & - & 7x_1 & & & + & 2x_3 & & \\ \text{subject to} & & & & & & & & \\ & & & - & 3x_2 & + & 4x_3 & \leq & 1, \\ & & x_1 & - & x_2 & & & \leq & 2, \\ & - & 3x_1 & & & + & x_3 & \leq & b, \\ & & & & & & x_1, x_2, x_3 & \geq & 0. \end{array} \tag{1}$$

(b)

What happens in (1) if $b = 0$ or $b = \frac{1}{4}$?

Problem 2

(a)

Consider the LP problem

$$\begin{array}{lrrrrcl} \text{maximize} & - & 3x_1 & + & 6x_2 & & \\ \text{subject to} & & & & & & \\ & & 2x_1 & + & x_2 & \leq & 6, \\ & - & x_1 & + & 2x_2 & \leq & 2, \\ & & & & x_1, x_2 & \geq & 0. \end{array} \tag{2}$$

Use the simplex algorithm to find *all* the optimal solutions.

(b)

Draw the feasible region of the LP problem (2) and highlight the optimal solutions in your drawing. Explain why the non-uniqueness of optimal solutions occurs.

(c)

Consider the LP problem

$$\begin{array}{lrrrcl} \text{maximize} & 3x_1 & + & 2x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & - & x_2 & \leq & 3, \\ & x_1 & & & \leq & 5, \\ & & & x_1, x_2 & \geq & 0. \end{array} \tag{3}$$

Show that (3) is unbounded by solving it via the simplex algorithm, and draw its feasible region.

Problem 3

In order to heal heart diseases it is essential to detect them early. In this assignment you are going to analyze how an imaginary heart disease detection system works. First, suppose each heart cell has a certain "level of disease", which is some real number between 0 (perfectly healthy cell) and 1 (completely corrupted cell). The goal is to detect the level of disease of each cell in the heart, in order to assign the patient a localized treatment. Our detector essentially consists in an Ω-ray machine, which shoots rays through the heart and keeps track of the total disease that each ray encounters, by summing up the level of disease of every cell along the path of the ray. The weakness of this method is that a ray can give information about the total quantity of disease it meets, but not *in which point* of its way the disease is located. However, the Ω-ray machine can shoot many rays, so that we hope to ultimately be able to reconstruct—at least partially—the disease-map of the heart.

To simplify the analysis, let us put ourselves into the weird setting of a 2-dimensional heart, having a discrete rectangular shape. That is, we view the heart as an $m \times n$ rectangular grid. Each cell of the grid corresponds then to a cell in the heart, so that we can define a *disease matrix* $D = \{d_{ij}\} \in \mathbb{R}^{m,n}$ whose (i, j)-th entry is the level of disease of the (i, j)-th cell of the heart.

We want each entry of D to be a real number between 0 and 1; that is,

$$0 \leq d_{ij} \leq 1, \quad i = 1, \ldots, m, \ j = 1, \ldots, n.$$

Suppose we shoot an horizontal Ω-ray through each row of the heart, and a vertical Ω-ray through each column. Given $i \in \{1, \ldots, m\}$, the i-th horizontal ray will send back to the detector an information corresponding to the total disease of the first row of D; i.e., the number

$$r_i = \sum_{j=1}^{n} d_{ij}.$$

Analogously, given $j \in \{1, \ldots, n\}$, the j-th vertical ray will send to the detector the number

$$c_j = \sum_{i=1}^{m} d_{ij}.$$

Clearly, the following equation must hold:

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j.$$

The goal is now to reconstruct D from the known arrays $r = (r_1, \ldots, r_m)$ and $c = (c_1, \ldots, c_n)$. NOTE: The matrix D we want to find has mn entries, while we are only given $m + n$ known quantities. This means that the matrix D cannot be uniquely determined just by r and c. Therefore, the problem is to find *one* of possibly many configurations for D.

(a)

Consider the following LP problem:

$$\text{maximize} \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} \tag{4}$$

subject to

$$\begin{aligned} 0 \le x_{ij} &\le 1, && i = 1, \dots, m,\ j = 1, \dots, n \\ \sum_{j=1}^{n} x_{ij} &\le r_i, && i = 1, \dots, m \\ \sum_{i=1}^{m} x_{ij} &\le c_j, && j = 1, \dots, n. \end{aligned}$$

Prove that the problem in the text has a solution if and only if (4) has an optimal solution $\hat{x} = \{\hat{x}_{ij}\}$ with corresponding objective function value s, where

$$s := \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j$$

and that in this case the matrix $\hat{X} := \{\hat{x}_{ij}\}$ is a feasible disease matrix.

(b)

Suppose that

$$\begin{aligned} m &= 5, \\ n &= 7, \\ r &= (3.2, 2.0, 2.8, 5.7, 3.3), \\ c &= (4.0, 2.1, 2.2, 3.2, 1.9, 1.3, 2.3). \end{aligned}$$

Implement an OPL-CPLEX algorithm to solve the LP problem (4) with the data given above. Does the solution you obtain correspond to a possible disease matrix? Explain why or why not.