20^{th} March, 2019

MAT3100 - SPRING 2019

Mandatory Assignment II

Submission deadline

Thursday 11^{th} April 2019 (upload to Devilry).

Instructions

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you understand the content of what you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code (and a trial run) along with the rest of the assignment.

Please write down the solution to the exercises in English.

All mandatory assignments must be uploaded to Devilry.

- The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. If these two requirements are not met, the assignment will not be assessed.
- The submission must contain your name, course and assignment number.
- *To have a passing grade you must have satisfactory answers to at least 50% of the questions and have seriously attempted to solve all of them. In particular, if you hand in a blank answer to any one of the questions, the assignment will not be approved.*

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you must contact the Student Administration at the Dep. of Math. (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1 (theory)

(a)

Let $A_1 \in \mathbb{R}^{m_1 \times n}$, $A_2 \in \mathbb{R}^{m_2 \times n}$, $b_1 \in \mathbb{R}^{m_1}$, $b_2 \in \mathbb{R}^{m_2}$, and $c \in \mathbb{R}^n$. Consider the LP problem

$$\max c \cdot x, \quad A_1 x \leq b_1, \quad A_2 x \geq b_2, \quad x \geq 0. \tag{1}$$

Determine the dual problem linked to (1).

(b)

(i) Identify the LP problem (P) that has the following problem as its dual:

$$\begin{aligned} &\min y_1 + 2y_2 + 3y_3, \\ &y_1 + y_2 \geq 4, \\ &y_2 + y_3 \geq 5, \\ &y_1 + y_3 \geq 6, \\ &y_1, y_2, y_3 \geq 0. \end{aligned} \tag{2}$$

(ii) State the complementary slackness theorem (for a general LP problem).

(iii) The optimal solution of (2) is $(y_1, y_2, y_3) = (\frac{5}{2}, \frac{3}{2}, \frac{7}{2})$. Use this and the complementary slackness theorem to solve problem (P).

(c)

Consider a standard LP problem

$$\begin{aligned} \max \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad i = 1, \ldots, m \\ & x_j \geq 0 \qquad j = 1, \ldots, n. \end{aligned}$$

Similarly to what we have been doing in the lectures (see pp. 81-82 of Vanderbei's book) we can write the problem in matrix form as

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{3}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & 1 & & & \\ a_{21} & a_{22} & \ldots & a_{2n} & & 1 & & \\ \ldots & \ldots & \ldots & \ldots & & & \ldots & \\ a_{m1} & a_{m2} & \ldots & a_{mn} & & & & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \\ 0 \\ \dots \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \\ x_{n+1} \\ \dots \\ x_{n+m} \end{pmatrix}$$

(notice that $x_{n+1}, \dots, x_{n+m}$ are the slack variables).

In an arbitrary step in the simplex algorithm, denote by $\mathcal{B}$ the set of indices corresponding to the basic variables, and by $\mathcal{N}$ the remaining nonbasic indices. Denote by $A_\mathcal{B}$ the $m \times m$ matrix whose columns consist of the m columns of A associated with the basic variables. Similarly, denote by $A_\mathcal{N}$ the $m \times n$ matrix whose columns are the n nonbasic columns of A. Let us write

$$A = \begin{bmatrix} A_\mathcal{B} & A_\mathcal{N} \end{bmatrix}, \quad c = \begin{pmatrix} c_\mathcal{B} \\ c_\mathcal{N} \end{pmatrix}, \quad x = \begin{pmatrix} x_\mathcal{B} \\ x_\mathcal{N} \end{pmatrix}.$$

(i) Show that the dictionary associated with the basis $\mathcal{B}$ can be written as

$$\begin{aligned} \eta &= \eta^* - z_\mathcal{N}^{*T} x_\mathcal{N}, \\ x_\mathcal{B} &= x_\mathcal{B}^* - A_\mathcal{B}^{-1} A_\mathcal{N} x_\mathcal{N}, \end{aligned}$$

where

$$\eta^* = c_\mathcal{B}^T A_\mathcal{B}^{-1} b, \quad z_\mathcal{N}^* = (A_\mathcal{B}^{-1} A_\mathcal{N})^T c_\mathcal{B} - c_\mathcal{N}, \quad x_\mathcal{B}^* = A_\mathcal{B}^{-1} b.$$

(ii) Consider the LP problem (3) with

$$c = (5, 4, 3, 0, 0, 0)^\top, \quad A = \begin{pmatrix} 2 & 3 & 1 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 2 & 0 & 0 & 1 \end{pmatrix}, \quad b = (5, 11, 8)^\top.$$

The optimal dictionary for this problem is

$$\begin{aligned} \max \eta &= 13 - 3x_2 - x_4 - x_6, \\ x_3 &= 1 + x_2 + 3x_4 - 2x_6, \\ x_1 &= 2 - 2x_2 - 2x_4 + x_6, \\ x_5 &= 1 + 5x_2 + 2x_4. \end{aligned} \tag{4}$$

Suppose the coefficient vector c is changed to

$$\bar{c} = (5, 4 + \varepsilon, 3, 0, 0, 0)^\top, \qquad \varepsilon > 0.$$

How large can ε be chosen without sacrificing the optimality of the solution

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (2, 0, 1, 0, 1, 0)?$$

Problem 2 (programming)

`480 BC, Greece`

The Persian king Xerxes is marching towards the free cities of Greece with a powerful army, determined to make the Hellenic peninsula just a province in his huge kingdom. The Greeks, for once allied under the command of Themistocles, have only one option left: resisting.

There are 3 hot-spots to defend: the narrow coastal pass of Thermopylae ("The Hot Gates"); the island of Salamin, near Athens; and the plain of Plataea, in Boeotia.

Battles take place during daytime; each night, the two leaders Xerxes and Themistocles decide how to distribute their battalions among the 3 locations for the next day fights. They must both deploy 5 battalions in whichever disposition they like, with the only constraint that each leader must place at least 1 battalion in every location. For example, if Themistocles thinks his best strategy is to overcome the Persians in the sea, he would probably place 1 battalion to protect Thermopylae, 3 battalions in the waters of Salamin, and the last 1 battalion in Plataea. Notice that neither Xerxes nor Themistocles know the strategy of the other leader in advance.

When the sun rises, the battles begin. The number of soldiers is very important in this kind of fights. This means that, if in a location one leader has placed more battalions than the other, he wins that battle. However, Greek warriors are slightly better trained and equipped than Persian ones, so that if the number of Persian and Greek battalions in a location is equal, the Greeks win.

(a)

Consider the problem as a matrix game. Xerxes is the row-player, Themistocles is the column-player. A pure strategy i for one of the two leaders is a triplet (i_1, i_2, i_3), where i_1, i_2 and i_3 are the numbers of battalions that the leader places in Thermopylae, Salamin and Plataea respectively.

Show that each leader can choose among 6 different pure strategies, which correspond to 6 different ways to distribute their own 5 battalions in the 3 locations.

(b)

Since each player has 6 pure strategies, we need to find a payoff matrix

$$A = [a_{ij}]_{i,j=1,\dots,6} \in \mathbb{R}^{6,6}.$$

Given a pure row strategy $i = (i_1, i_2, i_3)$ and a pure column strategy $j = (j_1, j_2, j_3)$, the corresponding payoff a_{ij} should express the overall result of the 3 battles in that specific day. Therefore, we set

$$a_{ij} = (\text{\# of battles won by the Greeks}) - (\text{\# of battles won by the Persians})$$

where the Persians are using strategy i ad the Greeks are using strategy j.
For example, suppose $i = (1, 2, 2)$ and $j = (1, 3, 1)$. Then:

- In Thermopylae there is 1 Persian battalion and 1 Greek battalion; the Greeks win.
- In Salamin there are 2 Persian battalions and 3 Greek battalions; the Greeks win.

- In Plataea there are 2 Persian battalions and 1 Greek battalion; the Persians win.

Therefore, we have $a_{ij} = 2 - 1 = 1$.

Write down the payoff matrix A. NOTE: this requires a lot of calculations. You could write an algorithm in whichever programming language you prefer to solve this task.

(c)

A mixed strategy for a player (Xerxes or Themistocles) is an array $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6$ of nonnegative numbers summing up to 1, where x_i expresses the probability for the player of playing strategy i. The goal is now to find an optimal column strategy $x^* \in \mathbb{R}^6$ for Themistocles, and an optimal row strategy $y^* \in \mathbb{R}^6$ for Xerxes. We know from the theory (see Chapter 11 of Vanderbei's book) that x^* is an optimal solution of the LP problem

$$\begin{aligned} &\text{maximize } v \\ &\text{subject to} \quad v \le e_i^T A x, \quad i = 1, \dots, 6, \\ &\sum_{j=1}^{6} x_j = 1, \\ &x_j \ge 0, \quad j = 1, \dots, 6 \end{aligned} \tag{5}$$

while y^* is an optimal solution of the LP problem

$$\begin{aligned} &\text{minimize } u \\ &\text{subject to} \quad u \ge e_j^T A^T y, \quad j = 1, \dots, 6, \\ &\sum_{i=1}^{6} y_i = 1, \\ &y_i \ge 0, \quad i = 1, \dots, 6. \end{aligned} \tag{6}$$

Use OPL-CPLEX, MATLAB or any programming language you prefer to solve problems (5) and (6), and find the optimal strategies that Xerxes and Themistocles should use in order to maximize their chances to win the Second Greco-Persian War. What is the value of this game?

Problem 3 (programming)

Consider the following LP problem in matrix form:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0. \end{aligned}$$

In this exercise you will program an algorithm to solve this problem, based on the **path-following method**, presented in Figure 18.1 of Vanderbei's book. To have more details, you will need to consult chapters 17-19 of the book. The

function should be called `interior`, take A, b and c as arguments (and perhaps a tolerance ϵ), and return an optimal solution x.
We will assume that $b \geq 0$. Your solution should contain the following:

- The code for the algorithm.
- A test run using the data files accompanying this assignment (print out a line of information for each iteration (or for every few iterations)).
- A brief discussion about the implementation of the algorithm.

You may choose whatever language you like, but we recommend MATLAB. Some tips:

- When solving a linear system $Vx = d$ do not use the `inv(V)` command (or the corresponding command in your language of choice). Rather, use the backslash operator: `x = V\d`.
- The major part of the work is solving the KKT-system to find the steps $\Delta x, \Delta y, \Delta z$ and Δw. There are various ways to do this, but we suggest the *normal equations in primal form*, see the start of section 19 in the book (equation (19.9)).
- To get the test data – A, b and c – you can:
  - use the command `load('data.mat')` if you are using MATLAB;
  - import numpy as `np` and use the commands

    ```
    A = np.loadtxt('A.txt', delimiter = ',')
    b = np.loadtxt('b.txt', delimiter = ',')
    c = np.loadtxt('c.txt', delimiter = ',')
    ```

    if you are using Python;
  - use a suitable parser if you are using a different programming language.
- The optimal solution value for the test problem is ≈ 321.4047.
- The text problem is quite large, so it may be a good idea to develop against a simpler problem at first.