Answers to Exercises, Week 13, MAT3100, V20

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Exercises in Week 13 are: 19-24 from 'A mini-introduction to convexity' by Geir Dahl.

Exercise 19

$$E = \{\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) : \epsilon_i \in \{a_i, b_i\}, i = 1, 2, \ldots, n\}.$$

Exercise 20

We want the extreme points of

$$P = \{x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1, x_2 \leq 2\}.$$

They are

$$E = \{(0,0), (1,0), (1/2, 1/2), (0, 1/2)\}.$$

Exercise 21

We want the extreme points of

$$P = \{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 4, x_1 + x_2 \geq 1, x_3 \leq 2\}.$$

They are

$$E = \{(4,0,0), (0,4,0), (0,1,0), (1,0,0), (2,0,2), (0,2,2), (0,1,2), (1,0,2)\}.$$

Exercise 22

Use Fourier-Motzkin elimination to solve the inequalities in Exercise 20.
Answer. The system of inequalities is

$$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0, \\ x_1 + x_2 &\leq 1, \\ x_2 &\leq 2. \end{aligned}$$

Gather all inequalities involving x_1:

$$0 \leq x_1 \leq 1 - x_2.$$

This implies

$$0 \leq 1 - x_2,$$

or

$$x_2 \leq 1,$$

and we add this to the remaining inequalities:

$$\begin{aligned} x_2 &\leq 1, \\ x_2 &\geq 0, \\ x_2 &\leq 2, \end{aligned}$$

and reduce:

$$0 \leq x_2 \leq 1.$$

Hence the solution is

$$\begin{aligned} &0 \leq x_1 \leq 1 - x_2, \\ &0 \leq x_2 \leq 1. \end{aligned}$$

Exercise 23

Use Fourier-Motzkin elimination to solve the inequalities in Exercise 21.
Answer. The system of inequalities is

$$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0, \\ x_3 &\geq 0, \\ x_1 + x_2 + x_3 &\leq 4, \\ x_1 + x_2 &\geq 1, \\ x_3 &\leq 2. \end{aligned}$$

Gather all inequalities involving x_1:

$$\max(0, 1 - x_2) \leq x_1 \leq 4 - x_2 - x_3.$$

This implies

$$\max(0, 1 - x_2) \leq 4 - x_2 - x_3.$$

or equivalently

$$\begin{aligned} 0 &\leq 4 - x_2 - x_3, \\ 1 - x_2 &\leq 4 - x_2 - x_3, \end{aligned}$$

or equivalently

$$\begin{aligned} x_2 + x_3 &\leq 4, \\ x_3 &\leq 3, \end{aligned}$$

and we add these to the remaining inequalities:

$$\begin{aligned} x_2 + x_3 &\leq 4, \\ x_3 &\leq 3, \\ x_2 &\geq 0, \\ x_3 &\geq 0, \\ x_3 &\leq 2. \end{aligned}$$

and reduce:

$$\begin{aligned} x_2 + x_3 &\leq 4, \\ x_2 &\geq 0, \\ x_3 &\geq 0, \\ x_3 &\leq 2. \end{aligned}$$

We now repeat the procedure, eliminating x_2:

$$0 \leq x_2 \leq 4 - x_3.$$

Thus implies

$$0 \leq 4 - x_3,$$

or equivalently

$$x_3 \le 4,$$

and we add this to the remaining inequalities:

$$\begin{aligned} x_3 &\le 4, \\ x_3 &\ge 0, \\ x_3 &\le 2. \end{aligned}$$

We then reduce:

$$0 \le x_3 \le 2.$$

Hence the solution is

$$\begin{aligned} \max(0, 1 - x_2) \le x_1 &\le 4 - x_2 - x_3, \\ 0 \le x_2 &\le 4 - x_3, \\ 0 \le x_3 &\le 2. \end{aligned}$$

Exercise 24

Use Fourier-Motzkin elimination in the case $n = 2$ and $m = 4$. Answer: The general system of inequalities is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &\le b_1, \\ a_{21}x_1 + a_{22}x_2 &\le b_2, \\ a_{31}x_1 + a_{32}x_2 &\le b_3, \\ a_{41}x_1 + a_{42}x_2 &\le b_4. \end{aligned}$$

Let $I^+ = \{i : a_{i1} > 0\}$, $I^0 = \{i : a_{i1} = 0\}$, $I^- = \{i : a_{i1} < 0\}$. Then the system is

$$\begin{aligned} x_1 &\le \frac{b_i}{a_{i,1}} - \frac{a_{i,2}}{a_{i,1}} x_2, && i \in I^+, \\ x_1 &\ge \frac{b_i}{a_{i,1}} - \frac{a_{i,2}}{a_{i,1}} x_2, && i \in I^-, \\ a_{i,2}x_2 &\le b_i, && i \in I^0. \end{aligned}$$

Therefore,

$$\max_{i \in I^-} \left(\frac{b_i}{a_{i,1}} - \frac{a_{i,2}}{a_{i,1}} x_2 \right) \le x_1 \le \min_{j \in I^+} \left(\frac{b_j}{a_{j,1}} - \frac{a_{j,2}}{a_{j,1}} x_2 \right). \tag{1}$$

This is the inequality for x_1. It remains to solve for x_2. Due to (1),

$$\max_{i \in I^-} \left(\frac{b_i}{a_{i,1}} - \frac{a_{i,2}}{a_{i,1}} x_2 \right) \leq \min_{j \in I^+} \left(\frac{b_j}{a_{j,1}} - \frac{a_{j,2}}{a_{j,1}} x_2 \right),$$

or equivalently

$$\frac{b_i}{a_{i,1}} - \frac{a_{i,2}}{a_{i,1}} x_2 \leq \frac{b_j}{a_{j,1}} - \frac{a_{j,2}}{a_{j,1}} x_2, \quad i \in I^-, \quad j \in I^+,$$

$$\left(\frac{a_{j,2}}{a_{j,1}} - \frac{a_{i,2}}{a_{i,1}} \right) x_2 \leq \frac{b_j}{a_{j,1}} - \frac{b_i}{a_{i,1}}, \quad i \in I^-, \quad j \in I^+,$$

Thus x_2 solves the system

$$\left(\frac{a_{j,2}}{a_{j,1}} - \frac{a_{i,2}}{a_{i,1}} \right) x_2 \leq \frac{b_j}{a_{j,1}} - \frac{b_i}{a_{i,1}}, \quad i \in I^-, j \in I^+,$$
$$a_{k,2} x_2 \leq b_k, \quad k \in I^0.$$