Answers to Exercises, Week 6, MAT3100, V20

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Exercises in Week 6 are: 5.1, 5.4, 5.5, 5.7 of Vanderbei.

Exercise 5.1

First we convert the problem to standard form. x_1 is a free variable and so we replace it by the difference $x_0 - x_1$ where $x_0, x_1 \geq 0$. Similarly, x_4 is a free variable and so we replace it by the difference $x_4 - x_5$ where $x_4, x_5 \geq 0$. Then we obtain

$$\begin{array}{lrl}
\text{maximize} & x_0 - x_1 - 2x_2 & \\
\text{subject to} & x_0 - x_1 + 2x_2 - x_3 + x_4 - x_5 & \geq 0, \\
& 4x_0 - 4x_1 + 3x_2 + 4x_3 - 2x_4 + 2x_5 & \leq 3, \\
& -x_0 + x_1 - x_2 + 2x_3 + x_4 - x_5 & = 1, \\
& x_0, \ldots, x_5 & \geq 0.
\end{array}$$

Then we invert the $\geq$ inequality and replace the equality by $\leq$ and $\geq$ and we get

$$\begin{array}{lrl}
\text{maximize} & x_0 - x_1 - 2x_2 & \\
\text{subject to} & -x_0 + x_1 - 2x_2 + x_3 - x_4 + x_5 & \leq 0, \\
& 4x_0 - 4x_1 + 3x_2 + 4x_3 - 2x_4 + 2x_5 & \leq 3, \\
& -x_0 + x_1 - x_2 + 2x_3 + x_4 - x_5 & \leq 1, \\
& x_0 - x_1 + x_2 - 2x_3 - x_4 + x_5 & \leq -1, \\
& x_0, \ldots, x_5 & \geq 0.
\end{array}$$

Then the dual problem is

$$\begin{array}{lrl}
\text{minimize} & 3y_1 + y_2 - y_3 & \\
\text{subject to} & -y_0 + 4y_1 - y_2 + y_3 & \geq 1, \\
& y_0 - 4y_1 + y_2 - y_3 & \geq -1, \\
& -2y_0 + 3y_1 - y_2 + y_3 & \geq -2, \\
& y_0 + 4y_1 + 2y_2 - 2y_3 & \geq 0, \\
& -y_0 - 2y_1 + y_2 - y_3 & \geq 0, \\
& y_0 + 2y_1 - y_2 + y_3 & \geq 0, \\
& y_0, \ldots, y_3 & \geq 0.
\end{array}$$

Exercise 5.4

The LP problem is

$$\begin{array}{rll} \text{maximize} & 2x_1 + x_2 & \\ \text{subject to} & 2x_1 + x_2 & \le 4, \\ & 2x_1 + 3x_2 & \le 3, \\ & 4x_1 + x_2 & \le 5, \\ & x_1 + 5x_2 & \le 1, \\ & x_1, x_2 & \ge 0. \end{array}$$

The dual problem is

$$\begin{array}{rll} \text{minimize} & 4y_1 + 3y_1 + 5y_4 + y_4 & \\ \text{subject to} & 2y_1 + 2y_2 + 4y_3 + y_4 & \ge 2, \\ & y_1 + 3y_2 + y_3 + 5y_4 & \ge 1, \\ & y_1, y_2, y_3, y_4 & \ge 0. \end{array}$$

We introduce slack variables for both problems. The two initial dictionaries are:

$$\begin{array}{rcrcrcr} \eta & = & & & 2x_1 & + & x_2 \\ \hline w_1 & = & 4 & - & 2x_1 & - & x_2 \\ w_2 & = & 3 & - & 2x_1 & - & 3x_2 \\ w_3 & = & 5 & - & 4x_1 & - & x_2 \\ w_4 & = & 1 & - & x_1 & - & 5x_2 \end{array}$$

and

$$\begin{array}{rcrcrcrcrcr} -\zeta & = & & - & 4y_1 & - & 3y_2 & - & 5y_3 & - & y_4 \\ \hline z_1 & = & -2 & + & 2y_1 & + & 2y_2 & + & 4y_3 & + & y_4 \\ z_2 & = & -1 & + & y_1 & + & 3y_2 & + & y_3 & + & 5y_4 \end{array}$$

The primal dictionary is feasible, the dual not. We pivot on the primal dictionary. x_1 enters and w_4 leaves (as in Exercise 2.2). The new dictionary is:

$$\begin{array}{rcrcrcr} \eta & = & 2 & - & 2w_4 & - & 9x_2 \\ \hline w_1 & = & 2 & + & 2w_4 & + & 9x_2 \\ w_2 & = & 1 & + & 2w_4 & + & 7x_2 \\ w_3 & = & 1 & + & 4w_4 & + & 19x_2 \\ x_1 & = & 1 & - & w_4 & - & 5x_2 \end{array}$$

Then we apply the corresponding pivot to the dual dictionary: i.e., z_1 leaves and y_4 enters. The new dictionary is:

$$
\begin{array}{rcrcrcrcrcr}
-\zeta & = & -2 & - & 2y_1 & - & y_2 & - & y_3 & - & z_1 \\
\hline
y_4 & = & 2 & - & 2y_1 & - & 2y_2 & - & 4y_3 & + & z_1 \\
z_2 & = & 9 & - & 9y_1 & - & 7y_2 & - & 19y_3 & + & 5z_1
\end{array}
$$

The primal dictionary is now optimal. and the dual dictionary is feasible. We also see that both objective functions have the same value. Thus, this illustrates the Strong Duality Theorem.

Exercise 5.5

(a) The dual problem is

$$
\begin{array}{ll}
\text{minimize} & 6y_1 + 1.5y_2 + 4y_3 \\
\text{subject to} & 2y_1 - 2y_2 + 3y_3 \geq 2, \\
 & 3y_1 + 4y_2 + 2y_3 \geq 8, \\
 & 3y_2 - 2y_3 \geq -1, \\
 & 6y_1 - 4y_3 \geq -2, \\
 & y_1, y_2, y_3 \geq 0.
\end{array}
$$

(b) The basic variables are x_1, w_2, x_3. The non-basic ones are w_1, x_2, w_3, x_4.
(c) The primal solution (basic solution in the primal problem) is $x_1 = 3.0$, $w_2 = 0.0$, $x_3 = 2.5$. It is feasible because all these basic variables have non-negative values. It is degenerate because $w_2 = 0$.
(d) We use the negative transpose property: the corresponding dual dictionary is

$$
\begin{array}{rcrcrcrcr}
-\zeta & = & -3.5 & - & 3.0z_1 & + & 0y_2 & - & 2.5z_3 \\
\hline
y_1 & = & 0.25 & + & 0.5z_1 & - & 1.25y_2 & + & 0.75z_3 \\
z_2 & = & -6.25 & + & 1.5z_1 & + & 3.25y_2 & + & 1.25z_3 \\
y_3 & = & 0.5 & + & 0z_1 & + & 1.5y_2 & - & 0.5z_3 \\
z_4 & = & 1.5 & + & 3.0z_1 & - & 13.5y_2 & + & 6.5z_3
\end{array}
$$

(e) The dual solution is $y_1 = 0.25$, $z_2 = -6.25$, $y_3 = 0.5$, $z_4 = 1.5$, and $z_1 = y_2 = z_3 = 0$. It is not feasible because z_2 is negative.
(f) Yes, the solutions satisfy the complementary slack conditions:

$$x_i z_i = 0, \quad i = 1, \ldots, 4, \qquad y_i w_i = 0, \quad i = 1, \ldots, 3.$$

Note y_2 and w_2 are both zero.

(g) No, the current primal solution is not optimal because x_2 has a positive coefficient, 6.25, in the objective function.
(h) In the next primal pivot, x_2 will enter and w_2 will leave. This is a degenerate pivot because w_2 has the current value 0. So the value of η will not change.

Exercise 5.7

The LP problem in Exercise 2.3 is

$$\begin{array}{ll} \text{maximize} & 2x_1 - 6x_2 \\ \text{subject to} & -x_1 - x_2 - x_3 \leq -2, \\ & 2x_1 - x_2 + x_3 \leq 1, \\ & x_1, x_2, x_3 \geq 0. \end{array}$$

The dual problem is

$$\begin{array}{ll} \text{minimize} & -2y_1 + y_2 \\ \text{subject to} & -y_1 + 2y_2 \geq 2, \\ & -y_1 - y_2 \geq -6, \\ & -y_1 + y_2 \geq 0, \\ & y_1, y_2 \geq 0. \end{array}$$

The initial primal dictionary is

$$\begin{array}{rcrcrcrcr} \eta & = & & & 2x_1 & - & 6x_2 & & \\ \hline w_1 & = & -2 & + & x_1 & + & x_2 & + & x_3 \\ w_2 & = & 1 & - & 2x_1 & + & x_2 & - & x_3 \end{array}$$

and the initial dual dictionary is

$$\begin{array}{rcrcrcr} -\zeta & = & & & 2y_1 & - & y_2 \\ \hline z_1 & = & -2 & - & y_1 & + & 2y_2 \\ z_2 & = & 6 & - & y_1 & - & y_2 \\ z_3 & = & 0 & - & y_1 & + & y_2 \end{array}$$

Both dictionaries are non-feasible. We can use the dual-primal two-phase algorithm to solve this. In Phase I we find a feasible solution to the primal problem. We do this by replacing the objective function by $\eta = -x_1 - x_2 - x_3$. This has the effect of making the dual problem feasible. We then solve the dual problem, and use the result as a feasible solution to the primal problem. With $\eta = -x_1 - x_2 - x_3$, the dual dictionary is now

$$\begin{array}{rcrcrcr}
-\zeta & = & & & 2y_1 & - & y_2 \\
\hline
z_1 & = & 1 & - & y_1 & + & 2y_2 \\
z_2 & = & 1 & - & y_1 & - & y_2 \\
z_3 & = & 1 & - & y_1 & + & y_2
\end{array}$$

which is feasible. Then y_1 enters and there is a tie for the leaving variable among z_1, z_2, z_3. Let's choose z_2 to leave because then the coefficient of y_2 will become negative. We get

$$\begin{array}{rcrcrcr}
-\zeta & = & 2 & - & 2z_2 & - & 3y_2 \\
\hline
z_1 & = & 0 & + & z_2 & + & 3y_2 \\
y_1 & = & 1 & - & z_2 & - & y_2 \\
z_3 & = & 0 & + & z_2 & + & 2y_2
\end{array}$$

This dictionary is optimal. The corresponding primal dictionary is

$$\begin{array}{rcrcrcrcr}
\eta & = & -2 & + & 0x_1 & - & w_1 & + & 0x_3 \\
\hline
x_2 & = & 2 & - & x_1 & + & w_1 & - & x_3 \\
w_2 & = & 3 & - & 3x_1 & + & w_1 & - & 2x_3
\end{array}$$

This gives us the feasible solution $x_1 = x_3 = 0$, $x_2 = 2$. We can check that this is indeed feasible. We now express the original objective function in terms of the current non-basic variables:

$$\begin{aligned}
\eta &= 2x_1 - 6x_2 \\
&= 2x_1 - 6(2 - x_1 + w_1 - x_3) \\
&= -12 + 8x_1 - 6w_1 + 6x_3.
\end{aligned}$$

So we can start with the feasible dictionary

$$\begin{array}{rcrcrcrcr}
\eta & = & -12 & + & 8x_1 & - & 6w_1 & + & 6x_3 \\
\hline
x_2 & = & 2 & - & x_1 & + & w_1 & - & x_3 \\
w_2 & = & 3 & - & 3x_1 & + & w_1 & - & 2x_3
\end{array}$$

This is exactly the same dictionary we had at the start of Phase II when we solved Exercise 2.3 (In Week 3). After two pivots we obtain the solution $x_1 = 0$, $x_2 = 1/2$, $x_3 = 3/2$ with $\eta = -3$. We also have $w_1 = w_2 = 0$.