Answers to Exercises, Week 7, MAT3100, V20

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Exercises in Week 7 are: 6.1, 6.6, 7.1 of Vanderbei.

Exercise 6.1

(a) The basic variables are x_3, x_1, the non-basic ones are x_4, x_2, x_5.
(b) Recall that we can express the current dictionary as

$$\begin{array}{rcccc} \eta & = & c_B^T A_B^{-1} b & - & (z_N^*)^T x_N \\ \hline x_B & = & A_B^{-1} b & - & A_B^{-1} A_N x_N \end{array}$$

where

$$z_N^* = (A_B^{-1} A_N)^T c_B - c_N.$$

In the example, $B = \{3, 1\}$ and $N = \{4, 2, 5\}$. So $x_B^* = (2, 0)$.
(c) $z_N^* = (z_4, z_2, z_5) = (3, -2, 0)$.
(d)

$$A_B^{-1} A_N = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 1 & -3 \end{bmatrix}.$$

(e) Yes, becasue $x_B^* \geq 0$.
(f) No, because $z_N^* \ngeq 0$.
(g) Yes, because $x_1 = 0$.

Exercise 6.6

The safest approach is to write the given problem in standard form. Then it's easy to find the dual (even though there might be some short cuts). For the given problem, we could let $\tilde{x} = x - l$, so that then $\tilde{x}$ is a non-negative variable. With this substitution the problem becomes

$$\begin{array}{ll} \text{maximize} & c^T l + c^T \tilde{x} \\ \text{subject to} & a - Al \leq A\tilde{x} \leq b - Al, \\ & \qquad\;\; 0 \leq \tilde{x} \leq u - l. \end{array}$$

Note that the term $c^T l$ is a constant and will not affect the optimal solution $\tilde{x}$, but will affect the value of the objective function. We can further write the problem as

$$\begin{array}{lrl} \text{maximize} & c^T l + c^T \tilde{x} & \\ \text{subject to} & A\tilde{x} & \leq b - Al \\ & -A\tilde{x} & \leq Al - a, \\ & \tilde{x} & \leq u - l, \\ & \tilde{x} & \geq 0. \end{array}$$

Now we can let

$$\tilde{A} = \begin{bmatrix} A \\ -A \\ I_n \end{bmatrix}, \qquad \tilde{b} = \begin{bmatrix} b - Al \\ Al - a \\ u - l \end{bmatrix},$$

so that the problem is

$$\begin{array}{lrl} \text{maximize} & c^T l + c^T \tilde{x} & \\ \text{subject to} & \tilde{A}\tilde{x} & \leq \tilde{b} \\ & \tilde{x} & \geq 0. \end{array}$$

Now we can write down the dual problem. We just need to retain the constant term $c^T l$:

$$\begin{array}{lrl} \text{minimize} & c^T l + \tilde{b}^T y & \\ \text{subject to} & \tilde{A}^T y & \geq c \\ & y & \geq 0. \end{array}$$

Exercise 7.1

(a) We use the fact that the final dictionary can be written in the form

$$\begin{array}{rcrcr} \eta & = & c_B^T A_B^{-1} b & - & (z_N^*)^T x_N \\ \hline x_B & = & A_B^{-1} b & - & A_B^{-1} A_N x_N \end{array}$$

where

$$z_N^* = (A_B^{-1} A_N)^T c_B - c_N.$$

Since $B = \{2, 3, 7\}$ and $N = \{1, 5, 6, 4\}$ and $c = [1, 2, 1, 1, 0, 0, 0]^T$ we have

$$\begin{aligned} c_B &= [c_2, c_3, c_7]^T = [2, 1, 0]^T, \\ c_N &= [c_1, c_5, c_6, c_4]^T = [1, 0, 0, 1]^T. \end{aligned}$$

We are asked what happens if the objective function is changed to $\tilde{c}^T x$, where

$$\tilde{c} = [3, 2, 1, 1, 0, 0, 0]^T.$$

The only change is in c_1 and we have

$$\tilde{c}_B = c_B, \qquad \tilde{c}_N = c_N + [2, 0, 0, 0]^T.$$

The only change to the dictionary is in the term z_N^*, and we get

$$\begin{aligned}
\tilde{z}_N^* &= (A_B^{-1} A_N)^T c_B - \tilde{c}_N \\
&= z_N^* - [2, 0, 0, 0]^T \\
&= [1.2, 0.2, 0.9, 2.8]^T - [2, 0, 0, 0]^T \\
&= [-0.8, 0.2, 0.9, 2.8]^T.
\end{aligned}$$

So the current solution is no longer optimal. The new dictionary is

$$\begin{array}{rcrcrcrcrcr}
\eta & = & 12.4 & + & 0.8x_1 & - & 0.2x_5 & - & 0.9x_6 & - & 2.8x_4 \\
\hline
x_2 & = & 6 & - & x_1 & & & - & 0.5x_6 & - & 2x_4 \\
x_3 & = & 0.4 & - & 0.2x_1 & - & 0.2x_5 & + & 0.1x_6 & + & 0.2x_4 \\
x_7 & = & 11.2 & - & 1.6x_1 & + & 0.4x_5 & + & 0.3x_6 & + & 1.6x_4
\end{array}$$

To find the optimal solution we need to proceed with the simplex algorithm. x_1 enters the basis and x_3 leaves and we get:

$$\begin{array}{rcrcrcrcrcr}
\eta & = & 14 & - & 4x_3 & - & x_5 & - & 0.5x_6 & - & 2x_4 \\
\hline
x_2 & = & 4 & + & 5x_3 & + & x_5 & - & x_6 & - & 3x_4 \\
x_1 & = & 2 & - & 5x_3 & - & x_5 & + & 0.5x_6 & + & x_4 \\
x_7 & = & 8 & + & 8x_3 & + & 2x_5 & - & 0.5x_6 & &
\end{array}$$

This dictionary is optimal and the solution is

$$x^* = [2, 4, 0, 0, 0, 0, 8]^T, \quad \eta^* = 14.$$

(b) Now we change c to

$$\tilde{c} = [1, 2, 0.5, 1, 0, 0, 0]^T.$$

Then the only change is in c_3 and we have

$$\tilde{c}_B = c_B + [0, -0.5, 0]^T, \qquad \tilde{c}_N = c_N.$$

The only change to the dictionary is in the objective function. We get

$$\begin{aligned}\tilde{z}_N^* &= (A_B^{-1}A_N)^T\tilde{c}_B - c_N \\ &= z_N^* + (A_B^{-1}A_N)^T[0,-0.5,0]^T.\end{aligned}$$

From the given dictionary,

$$A_B^{-1}A_N = \begin{bmatrix} 1 & 0 & 0.5 & 2 \\ 0.2 & 0.2 & -0.1 & -0.2 \\ 1.6 & -0.4 & -0.3 & -1.6 \end{bmatrix},$$

and therefore

$$(A_B^{-1}A_N)^T[0,-0.5,0]^T = \begin{bmatrix} 1 & 0.2 & 1.6 \\ 0 & 0.2 & -0.4 \\ 0.5 & -0.1 & -0.3 \\ 2 & -0.2 & -1.6 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.1 \\ -0.1 \\ 0.05 \\ 0.1 \end{bmatrix}.$$

Also, from the dictionary,

$$z_N^* = [1.2, 0.2, 0.9, 2.8]^T,$$

and so

$$\tilde{z}_N^* = [1.2, 0.2, 0.9, 2.8]^T + [0.1, -0.1, 0.05, 0.1]^T = [1.1, 0.1, 0.95, 2.9]^T.$$

Hence, $\tilde{z}_N^* \geq 0$ and so x^* remains optimal. We can also find the new objective value:

$$\begin{aligned}\tilde{\eta}^* &= \tilde{c}_B^T A_B^{-1} b \\ &= \eta^* + [0,-0.5,0]A_B^{-1}b \\ &= 12.4 + [0,-0.5,0][6, 0.4, 11.2]^T \\ &= 12.4 - 0.2 \\ &= 12.2.\end{aligned}$$

(c) Now we change $b = [8, 12, 18]^T$ to $\tilde{b} = [8, 26, 18]^T$. The change to b is b_2 and

$$\tilde{b} = b + [0, 14, 0]^T.$$

Let's compute the modified dictionary. We can do this by computing the inverse of A_B. Since

$$A = \begin{bmatrix} 2 & 1 & 5 & 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_B = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix},$$

and we find

$$A_B^{-1} = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.2 & -0.1 & 0 \\ -0.4 & -0.3 & 1 \end{bmatrix}.$$

Then

$$\begin{aligned} A_B^{-1}\tilde{b} &= A_B^{-1}b + A_B^{-1}[0, 14, 0]^T \\ &= [6, 0.4, 11.2]^T + A_B^{-1}[0, 14, 0]^T \\ &= [6, 0.4, 11.2]^T + [7, -1.4, -4.2]^T \\ &= [13, -1, 7]^T, \end{aligned}$$

and

$$c_B^T A_B^{-1}\tilde{b} = [2, 1, 0][13, -1, 7]^T = 25.$$

The primal solution is no longer feasible, but the dual is. The updated dual dictionary is

$$\begin{array}{rcrcrcrcr} -\zeta & = & -25 & - & 13z_2 & + & z_3 & - & 7z_7 \\ \hline z_1 & = & 1.2 & + & z_2 & + & 0.2z_3 & + & 1.6z_7 \\ z_5 & = & 0.2 & & & + & 0.2z_3 & - & 0.4z_7 \\ z_6 & = & 0.9 & + & 0.5z_2 & - & 0.1z_3 & - & 0.3z_7 \\ z_4 & = & 2.8 & + & 2z_2 & - & 0.2z_3 & - & 1.6z_7 \end{array}$$

So we can now proceed with the simplex algorithm for the dual. z_3 enters the basis and z_6 leaves and we get:

$$\begin{array}{rcrcrcrcr} -\zeta & = & -16 & - & 8z_2 & - & 10z_6 & - & 10z_7 \\ \hline z_1 & = & 3 & + & 2z_2 & - & 2z_6 & + & z_7 \\ z_5 & = & 2 & + & z_2 & - & 2z_6 & - & z_7 \\ z_3 & = & 9 & + & 5z_2 & - & 10z_6 & - & 3z_7 \\ z_4 & = & 1 & + & z_2 & + & 2z_6 & - & z_7 \end{array}$$

This is optimal. Going to the primal dictionary, we obtain the solution

$$x^* = [0, 8, 0, 0, 0, 10, 10]^T, \quad \eta^* = 16.$$