Answers to Exercises, Week 8, MAT3100, V20

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Exercises in Week 8 are: 17.1, 17.2, 17.3, 17.4 of Vanderbei.

Exercise 17.1

The given problem is

$$\begin{array}{ll} \text{maximize} & -x_1 + x_2 \\ \text{subject to} & \qquad\qquad x_2 \le 1, \\ & -x_1 \qquad\quad \le -1, \\ & \qquad x_1, x_2 \ge 0. \end{array}$$

The optimal solution is clearly $(x_1, x_2) = (1, 1)$. The dual problem is

$$\begin{array}{ll} \text{minimize} & y_1 - y_2 \\ \text{subject to} & \qquad\quad -y_2 \ge -1, \\ & y_1 \qquad\quad \ge 1, \\ & \qquad y_1, y_2 \ge 0. \end{array}$$

We can rewrite this as

$$\begin{array}{ll} \text{maximize} & -y_1 + y_2 \\ \text{subject to} & \qquad\qquad y_2 \le 1, \\ & -y_1 \qquad\quad \le -1, \\ & \qquad y_1, y_2 \ge 0, \end{array}$$

and so we see that (D) equals (P).

The central path is the solution, for each $\mu > 0$, to the $2m+2n$ equations

$$\begin{aligned} Ax + w &= b, \\ A^T y - z &= c, \\ x_j z_j &= \mu, \quad \text{all } j, \\ w_i y_i &= \mu, \quad \text{all } i. \end{aligned}$$

Since (P) and (D) are the same we have $y_i = x_i$ and $z_i = w_i$ for all i, and therefore it is sufficient to solve

$$\begin{aligned} Ax + w &= b, \\ x_j w_j &= \mu, \quad \text{all } j. \end{aligned}$$

We have

$$c = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and so the equations become

$$\begin{aligned} x_2 + w_1 &= 1, \\ -x_1 + w_2 &= -1, \\ x_1 w_1 = x_2 w_2 &= \mu. \end{aligned}$$

Now we can eliminate w_1 and w_2 using the first two equations,

$$w_1 = 1 - x_2, \qquad w_2 = x_1 - 1,$$

to get

$$x_1(1 - x_2) = x_2(x_1 - 1) = \mu.$$

The first equation is the equation of a parabola. So the central path in the (x_1, x_2) plane is a parabola. However, we want the path as a function of μ. To do this we can eliminate x_2 using the second equation,

$$x_2 = \mu/(x_1 - 1)$$

and substitute into the first:

$$x_1(1 - \mu/(x_1 - 1)) = \mu,$$

or

$$x_1^2 - (1 + 2\mu)x_1 + \mu = 0.$$

The solution is

$$x_1 = (1 + 2\mu \pm \sqrt{1 + 4\mu^2})/2.$$

Since we must have $x_1 \geq 1$ for a feasible solution, we must have

$$x_1 = (1 + 2\mu + \sqrt{1 + 4\mu^2})/2.$$

A similar calculation gives

$$x_2 = (1 - 2\mu + \sqrt{1 + 4\mu^2})/2.$$

We find that the path $p(\mu) = (x_1(\mu), x_2(\mu))$ converges to $(x_1, x_2) = (1, 1)$ as $\mu \to 0$, which we can easily see is the optimal solution.

Exercise 17.2

The given problem is

$$\begin{array}{lrl} \text{maximize} & (\cos\theta)x_1 + (\sin\theta)x_2 & \\ \text{subject to} & x_1 \qquad\quad & \le 1, \\ & x_2 & \le 1, \\ & x_1, x_2 & \ge 0. \end{array}$$

The optimal solution is clearly $(x_1, x_2) = (1, 1)$. We have

$$c = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and the central path equations are

$$\begin{aligned} x_1 + w_1 &= 1, \\ x_2 + w_2 &= 1, \\ y_1 - z_1 &= \cos\theta, \\ y_2 - z_2 &= \sin\theta, \\ x_1 z_1 = x_2 z_2 = w_1 y_1 = w_2 y_2 &= \mu. \end{aligned}$$

In this case there are four equations in the unknowns x_1, w_1, y_1, z_1, and there are four other equations in the remaining variables. Thus it is sufficient to solve the first four:

$$\begin{aligned} x_1 + w_1 &= 1, \\ y_1 - z_1 &= \cos\theta, \\ x_1 z_1 = y_1 w_1 &= \mu. \end{aligned}$$

Using the last two, the first two become

$$\begin{aligned} x_1 + w_1 &= 1, \\ (1/w_1 - 1/x_1)\mu &= \cos\theta. \end{aligned}$$

Using the first to eliminate w_1 we find

$$(1/(1 - x_1) - 1/x_1)\mu = \cos\theta.$$

Let $\lambda = \cos\theta/\mu$. Then

$$\lambda x_1^2 + (2-\lambda)x_1 - 1 = 0,$$

and so

$$x_1 = (\lambda - 2 + \sqrt{\lambda^2 + 4})/(2\lambda)$$

(there is only one solution by the constraint that $x_1 \geq 0$). We can rewrite this as

$$x_1 = (c - 2\mu + \sqrt{c^2 + 4\mu^2})/(2c),$$

where $c = \cos\theta$. A similar calculation gives

$$x_2 = (c - 2\mu + \sqrt{c^2 + 4\mu^2})/(2c),$$

where $c = \sin\theta$. We can immediately see that $(x_1, x_2) \to (1, 1)$ as $\mu \to 0$. To get the limit as $\mu \to \infty$, we can rewrite x_1 as

$$\frac{2\mu}{-c + 2\mu + \sqrt{c^2 + 4\mu^2}},$$

and then we see that $x_1 \to 1/2$ as $\mu \to \infty$. Similarly, $x_2 \to 1/2$ as $\mu \to \infty$.

Exercise 17.3

We form the more general barrier problem

$$\begin{array}{ll} \text{maximize} & c^T x + \sum_j r_j \log x_j + \sum_i s_i \log w_i \\ \text{subject to} & Ax + w = b, \end{array}$$

for positive r_j and s_i. We now follow the same steps as in Chapter 17, using Lagrange multipliers and taking partial derivatives, and we end up with the four equations

$$\begin{aligned} Ax + w &= b, \\ A^T y - z &= c, \\ x_j z_j &= r_j, \quad \text{all } j, \\ w_i y_i &= s_i, \quad \text{all } i. \end{aligned}$$

The proof of existence and uniqueness are similar to Chapter 17.

Exercise 17.4

The given problem is

$$\begin{array}{lll} \text{maximize} & \sum_j c_j x_j & \\ \text{subject to} & \sum_j a_{ij} x_j & = b_i, \\ & x & \geq 0. \end{array}$$

Since the constraints are equalities, we could consider trying to solve

$$\begin{array}{ll} \text{maximize} & \sum_j c_j \xi_j^2 \\ \text{subject to} & \sum_j a_{ij} \xi_j^2 = b_i, \end{array}$$

with $\xi_1, \ldots, \xi_n$ free variables. If $\xi^* = (\xi_1^*, \ldots, \xi_n^*)$ is an optimal solution to the auxiliarly problem then $x^* = (\xi_1^2, \ldots, \xi_n^2)$ solves the original problem (it doesn't matter about the signs of $\xi_1^*, \ldots, \xi_n^*$).

The advantage of the auxiliary problem is that there are no inequalities and we could apply Lagrange multipliers. However, the problem is non-linear and might not be easy to solve.