Answers to Exercises, Week 9, MAT3100, V20

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Exercises in Week 9 are: 11.3, 11.2, 11.5, 11.6.

Exercise 11.3

Let's just answer 11.3(b) since 11.3(a) is similar. We suppose that in a matrix game, the payoff matrix $A \in \mathbb{R}^{m,n}$ has the property that column s is dominated by column r, i.e., that $a_{ir} \geq a_{is}$ for all $i = 1, \ldots, m$.

Let $x \in \mathbb{R}^n$ be a randomized strategy for the column player, i.e., $x \geq 0$ and $\sum_{j=1}^n x_j = 1$. Then

$$\begin{aligned} \min_i \sum_j a_{ij} x_j &= \min_i \left(\sum_{j \neq r,s} a_{ij} x_j + a_{ir} x_r + a_{is} x_s \right) \\ &\leq \min_i \left(\sum_{j \neq r,s} a_{ij} x_j + a_{ir} (x_r + x_s) \right) \\ &= \min_i \sum_{j \neq s} a_{ij} y_r, \end{aligned}$$

where $y_j = x_j$, $j \neq r$, and $y_r = x_r + x_s$. Then

$$y = (y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_n)$$

is a randomized strategy for the column player in the modified matrix game with payoff matrix B formed by removing column s from A. Therefore,

$$\min_i \sum_{j \neq s} a_{ij} y_r \leq \min_i \sum_{j \neq s} a_{ij} y_r^*,$$

where

$$y^* = (y_1^*, \ldots, y_{s-1}^*, y_{s+1}^*, \ldots, y_n^*)$$

is an optimal randomized strategy for the column player in the modified game. Then we conclude that

$$\min_i \sum_j a_{ij} x_j \leq \min_i \sum_{j \neq s} a_{ij} y_r^*,$$

and so

$$x^* = (y_1^*, \ldots, y_{s-1}^*, 0, y_{s+1}^*, \ldots, y_n^*)$$

is an optimal randomized strategy for the column player in the original game.

Now consider the payoff matrix

$$\begin{bmatrix} -6 & 2 & -4 & -7 & -5 \\ 0 & 4 & -2 & -9 & -1 \\ -7 & 3 & -3 & -8 & -2 \\ 2 & -3 & 6 & 0 & 3 \end{bmatrix}.$$

Since column 4 is dominated by column 3 we can remove column 4:

$$\begin{bmatrix} -6 & 2 & -4 & -5 \\ 0 & 4 & -2 & -1 \\ -7 & 3 & -3 & -2 \\ 2 & -3 & 6 & 3 \end{bmatrix}.$$

Now row 2 dominates row 1, and we can remove row 2 (from the row player's point of view):

$$\begin{bmatrix} -6 & 2 & -4 & -5 \\ -7 & 3 & -3 & -2 \\ 2 & -3 & 6 & 3 \end{bmatrix}.$$

Now column 1 is dominated by column 3 and we can remove column 1:

$$\begin{bmatrix} 2 & -4 & -5 \\ 3 & -3 & -2 \\ -3 & 6 & 3 \end{bmatrix}.$$

Now row 2 dominates row 1, and we can remove row 2:

$$\begin{bmatrix} 2 & -4 & -5 \\ -3 & 6 & 3 \end{bmatrix}.$$

Now column 3 is dominated by column 2 and we can remove column 3:

$$\begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}.$$

Exercise 11.2

The payoff matrix for this game is

$$\begin{bmatrix} 0 & 1 & -1 & -1 & -1 & -1 & \cdots \\ -1 & 0 & 1 & -1 & -1 & -1 & \cdots \\ 1 & -1 & 0 & 1 & -1 & -1 & \cdots \\ 1 & 1 & -1 & 0 & 1 & -1 & \cdots \\ 1 & 1 & 1 & -1 & 0 & -1 & \cdots \\ 1 & 1 & 1 & 1 & -1 & 0 & \cdots \\ \vdots & & & & & & \ddots \end{bmatrix}.$$

Then all columns 4 to 100 are dominated by the first column, and all rows 4 to 100 dominate row 1 and so the optimal strategies are the same as those for the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

This we recognise as the paper-scissors-rock game. So the optimal strategy for the column player is

$$x^* = (1/3, 1/3, 1/3, 0, \dots, 0) \in \mathbb{R}^{100},$$

and for the row player,

$$y^* = (1/3, 1/3, 1/3, 0, \dots, 0) \in \mathbb{R}^{100}.$$

Exercise 11.5

If the r-th pure row strategy and the s-th pure column strategy are simultaneously optimal then

$$\max_x (Ax)_r = \min_y (A^T y)_s,$$

and this is equivalent to

$$\max_{j=1,\dots,n} a_{rj} = \min_{i=1,\dots,m} a_{is}.$$

Exercise 11.6

Let $f(x) = \min_y y^T Ax$ and $g(y) = \max_x y^T Ax$. We know that the problems of maximizing $f(x)$ and minimizing $g(y)$ are dual problems. So, by the Weak Duality Theorem, $f(x) \leq g(y)$ for all feasible x and y. By the Minimax Theorem, there exist feasible x^* and y^* such that

$$\max_x (y^*)^T Ax = \min_y y^T Ax^*.$$

Therefore, $g(y^*) = f(x^*)$, and hence

$$\max_x f(x) = f(x^*) = g(y^*) = \min_y g(y).$$