Exercises, Week 9. Matrix Games

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

row player choose $i \in \{1, 2, 3\}$
column " " $j \in \{1, 2, 3\}$
Then row player pays a_{ij} to column player.

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -3 & 0 & 4 \end{bmatrix}.$$

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Column player chooses a prob. distribution $x = (x_1, \dots, x_n)$, $\sum_{i=1}^{n} x_i = 1$

$x_1, \dots, x_n \geq 0$. Row player choose y

payoff $= y^T A x$ (expected)

Column player: $\max_x \min_y y^T A x$

LP problem.

Simplification $\min_y y^T A x = \min_i e_i^T A x$

$= \min (Ax)$;

i

Row player:

$$\min_y \max_x y^T A x$$

$$= \min_y \max_j (y^T A)_j$$

Dual problem.

Strong Duality Theorem $\Rightarrow$ the two optimal values are same.

v = Optimal value = pay off of the game.

Cases

(i) $\eta > 0$ column player wins

(ii) $\eta = 0$ fair game

(iii) $\eta < 0$ row player wins.

Exercise 11.3.

$A \in \mathbb{R}^{m,n}$. Suppose column s is dominated by column r :: means

$$a_{ir} \geq a_{is} \quad \text{for all } i = 1 \ldots m.$$

Then show $x_s^* = 0$.

So can throw away column s.

$$\min_i \sum_j a_{ij} x_j = \min_i \left(\sum_{j \neq r, s} a_{ij} x_j + a_{ir} x_r + a_{is} x_s \right)$$

$$\leq \min_i \left(\sum_{j \neq r, s} a_{ij} x_j + a_{ir} (x_r + x_s) \right)$$

$$= \min_i \left(\sum_{j \neq s} a_{ij} y_j \right),$$

$y_j = x_j$, $j \neq r$, $y_r = x_r + x_s$.

Then $y = (y_1 \ldots y_{s-1}, y_{s+1} \ldots y_n)$

is a rand. strat. for B,

formed by removing column s

obtained by removing column s

from A. Let y^* opt. for B.

$$\min_i \sum a_{ij} y_j \leq \min_i \sum a_{ij} y_j^*$$

So $\min_i (Ax)_i \leq \min_i (By)_i$.

So $x^* = (y_1^* \dots y_{s-1}^*, 0, y_{s+1}^* \dots y_n^*)$

is optimal for A.

<u>Conclude</u>: can remove a column

if $\leq$ some other column.

can remove a row if

$\geq$ same row.

$a_4 \leq a_3$

Consider

$$A = \begin{bmatrix} -6 & 2 & -4 & -7 & -5 \\ 0 & 4 & -2 & -9 & -1 \\ -7 & 3 & -3 & -8 & -2 \\ 2 & -3 & 6 & 0 & 3 \end{bmatrix}$$

row 2 $\geq$ row 1

$$\begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$$

col play, $x^* = (0, x_2^*, x_3^*, 0, 0)$

row. $y^* = (y_1^*\ 0, 0, y_4^*)$

Exercise 11.2

$$A = \begin{bmatrix} 0 & 1 & -1 & -1 & -1 & \cdots \\ -1 & 0 & 1 & -1 & -1 & \\ 1 & -1 & 0 & 1 & -1 & \\ 1 & 1 & -1 & 0 & 1 & \\ 1 & 1 & 1 & -1 & 0 & \\ \vdots & & & & & \end{bmatrix} \in \mathbb{R}^{100,100}$$

L

Columns 4–100 dominated by col 1.
Rows 4–100 dominate row 1

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$ paper rock scissors.

$\hat{x}^* = (1/3, 1/3, 1/3)$ opt for B

$\hat{y}^* = (1/3, 1/3, 1/3)$.

$\Rightarrow \quad x^* = (1/3, 1/3, 1/3, 0, ---, 0) \in \mathbb{R}^{100}$

$y^* = (1/3\ 1/3\ 1/3,\ 0\ ---\ 0)$ "

How to solve LP ?

Column player.

$$\max_{x} \min_{i} (Ax)_i .$$

subj. $\sum x_j = 1$, $x_j \geq 0$.

$$\Leftrightarrow \quad \max \; v$$

subj $\quad v \leq (Ax)_i \;, \; i = 1, \ldots, m$

$$\sum x_j = 1$$

$$x_1, \ldots x_j \geq 0$$

Variables $x_1 \quad x \quad v$

variables $x_1, \dots, x_n, v$

$m+1$ constraints

$x_1 \dots x_n \geq 0, \quad v \in \mathbb{R}$

Standard form?

$$v = v_+ - v_-, \quad v_-, v_+ \geq 0$$

$$\max \quad v_+ - v_-$$

$$\text{subj} \quad v_+ - v_- \leq (Ax)_i = \sum_j a_{ij} x_j$$

$$\sum x_j \leq 1, \quad -\sum x_j \leq -1,$$

$$x_1 \dots x_n, v_-, v_+ \geq 0$$

$n+2$ variables

$m+2$ constraints.

Exercise 11.5

Pure strategy: always choose same row (row player)

payoff $\max_{x} (Ax)_r = \min_{y} (A^T y)_s$

$$\Longleftrightarrow \quad \max_{j=1..n} a_{rj} = \min_{i=1..m} a_{is}$$

Exercise 11.6

Use Minimax Theorem to show

$$\max_{x} \min_{y} y^T A x = \min_{y} \max_{x} y^T A x.$$

$$f(x) = \min_{y} y^T A x$$

$$g(y) = \max_{y} y^T A x$$

max. f dual min g.

W.D.T. $f(x) \le g(y)$, $\Rightarrow$ $\max_x f(x) \le \min_y g(y)$
all feasible x, y.

Minimax: $\exists$ feesible x_*, y_*

s.t. $\max_{x} (y^*)^T A x = \min_{y} y^T A x^*$

This implies $g(y^*) = f(x^*)$.

So no gap:

$$\max_x f(x) = \min_y g(y)$$