Answers to Compulsory Assignment 1
MAT3100 Linear Optimization, Spring 2020

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Problem 1

1a)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -3 & 4 \\ 1 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} -7 \\ 0 \\ 2 \end{bmatrix}.$$

1b)

The initial dictionary is

$$\begin{array}{rcrcrcrcr} \eta & = & & & & - & 7x_2 & + & 2\mathbf{x_3} \\ \hline w_1 & = & 1 & & & + & 3x_2 & - & 4x_3 \\ w_2 & = & 2 & - & x_1 & + & x_2 & & \\ \mathbf{w_3} & = & 0 & + & 3x_1 & & & - & x_3 \end{array}$$

x_3 enters, w_3 leaves:

$$\begin{array}{rcrcrcrcr} \eta & = & & - & x_1 & + & 0x_2 & - & 2w_3 \\ \hline w_1 & = & 1 & - & 12x_1 & + & 3x_2 & + & 4w_3 \\ w_2 & = & 2 & - & x_1 & + & x_2 & & \\ x_3 & = & 0 & - & 3x_1 & & & - & w_3 \end{array}$$

This is an optimal dictionary, so the optimal value is $\eta = 0$ and one solution point is $(x_1, x_2, x_3) = (0, 0, 0)$. We can obtain other solution points by observing that we can increase x_2 arbitrarily since in the x_2 column of the dictionary we have a 0 coefficient at the top, and non-negative coefficients below. So any point $(x_1, x_2, x_3) = (0, \lambda, 0)$ with $\lambda \geq 0$ is also a solution point.

1c)

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n+m} \end{bmatrix}, \; A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} & 1 & 0 & \cdots & 0 \\ a_{2,1} & \cdots & a_{2,n} & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & & \vdots \\ a_{m,1} & \cdots & a_{m,n} & 0 & 0 & \cdots & 1 \end{bmatrix}, \; b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \; c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Problem 2

2a)

The initial dictionary is

$$\begin{array}{rcrcrcr} \eta & = & & - & 3x_1 & + & 6x_2 \\ \hline w_1 & = & 6 & - & 2x_1 & - & x_2 \\ w_2 & = & 2 & + & x_1 & - & 2x_2 \end{array}$$

x_2 enters, w_2 leaves:

$$\begin{array}{rcrcrcr} \eta & = & 6 & + & 0x_1 & - & 3w_2 \\ \hline w_1 & = & 5 & - & (5/2)x_1 & + & (1/2)w_2 \\ x_2 & = & 1 & + & (1/2)x_1 & - & (1/2)w_2 \end{array}$$

This is an optimal dictionary. So $(x_1, x_2) = (0, 1)$ is an optimal solution. To find more optimal solutions, notice that the coefficent of x_1 in η in this dictionary is 0. So we can pivot again, letting x_1 enter the basis. Then w_1 leaves the basis and we get

$$\begin{array}{rcrcrcr} \eta & = & 6 & + & 0w_1 & - & 3w_2 \\ \hline x_1 & = & 2 & - & (2/5)w_1 & + & (1/5)w_2 \\ x_2 & = & 2 & - & (1/5)w_1 & - & (2/5)w_2 \end{array}$$

This is also an optimal dictionary and so $(x_1, x_2) = (2, 2)$ is another optimal solution. Therefore, all points of the form

$$(x_1, x_2) = (1 - \lambda)(0, 1) + \lambda(2, 2), \quad \lambda \in [0, 1],$$

are optimal solutions and $\eta = 6$ at all those points.

2b)

The feasible region is shown in Figure 1.

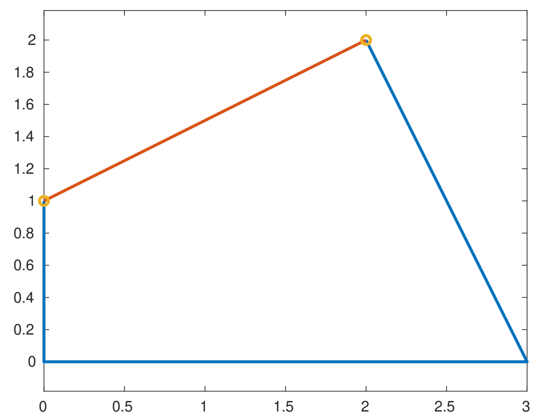

Figure 1: Feasible region and optimal solutions in red.

The non-uniqueness occurs because the vector $c = (-3, 6)$ defining the objective function $\eta = c_1 x_1 + c_2 x_2$ is perpendicular to the edge of the feasible region connecting $(0, 1)$ to $(2, 2)$:

$$(-3, 6) \cdot ((2, 2) - (0, 1)) = (-3, 6) \cdot (2, 1) = -3 \times 2 + 6 \times 1 = 0.$$

2c)

The initial dictionary is

$$\begin{array}{rcrcrcr} \eta & = & & + & 3x_1 & + & 2x_2 \\ \hline w_1 & = & 3 & - & x_1 & + & x_2 \\ w_2 & = & 2 & - & x_1 & & \end{array}$$

We could let x_2 enter the basis, but then we see that we can increase x_2 arbitrarily (keeping both w_1 and w_2 non-negative). So all points $(x_1, x_2) = (0, \lambda)$, where $\lambda \geq 0$, are feasible and $\eta = 2\lambda \to \infty$ as $\lambda \to \infty$. So the problem is unbounded.

Problem 3 – Linear Regression

3a)

We introduce the non-negative variables

$$t_i := |b_i - \sum_j a_{ij}x_j|, \quad i = 1, \ldots, n.$$

We then want to minimize $\sum_i t_i$. Since

$$t_i = \max\left(b_i - \sum_j a_{ij}x_j, \sum_j a_{ij}x_j - b_i\right),$$

we have the constraints

$$b_i - \sum_j a_{ij}x_j \le t_i, \qquad \sum_j a_{ij}x_j - b_i \le t_i,$$

or

$$-\sum_j a_{ij}x_j - t_i \le -b_i, \qquad \sum_j a_{ij}x_j - t_i \le b_i.$$

3b)

To solve the L_1 regression, we replace the free variables x_1 and x_2 by $y_1 - y_2$ and $y_3 - y_4$ respectively where $y_1, y_2, y_3, y_4 \ge 0$. We now have an LP problem in standard form:

$$\begin{array}{rrl}
\text{maximize} & -\sum_{i=1}^n t_i & \\
\text{subject to} & -a_1(y_1 - y_2) - (y_3 - y_4) - t_1 & \le -b_1, \\
& a_1(y_1 - y_2) + (y_3 - y_4) - t_1 & \le b_1, \\
& \vdots & \\
& -a_n(y_1 - y_2) - (y_3 - y_4) - t_n & \le -b_n, \\
& a_n(y_1 - y_2) + (y_3 - y_4) - t_n & \le b_n, \\
& y_1, y_2, y_3, y_4, t_1, \ldots, t_n & \ge 0.
\end{array}$$

We can write this as

$$\max c^T x \quad \text{subject to} \quad \tilde{A}x \le \tilde{b},\ x \ge 0,$$

where

$$x = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ t_1 \\ \vdots \\ t_n \end{bmatrix}, \tilde{b} = \begin{bmatrix} -b_1 \\ b_1 \\ -b_2 \\ b_2 \\ \vdots \end{bmatrix}, c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ \vdots \\ -1 \end{bmatrix},$$

and

$$\tilde{A} = \begin{bmatrix} -a_1 & a_1 & -1 & 1 & -1 & 0 & \cdots & 0 \\ a_1 & -a_1 & 1 & -1 & -1 & 0 & \cdots & 0 \\ -a_2 & a_2 & -1 & 1 & 0 & -1 & \cdots & 0 \\ a_2 & -a_2 & 1 & -1 & 0 & -1 & \cdots & 0 \\ \vdots & & & & & & & \end{bmatrix}.$$

I used the routine 'simplex.m' to solve this. The solution is

$$(y_1, y_2, y_3, y_4, t_1, t_2, \ldots) = (2, 0, 0, 0, *, *, \ldots).$$

Converting to the original variables we get

$$(x_1, x_2) = (y_1 - y_2, y_3 - y_4) = (2, 0).$$

The L_2 regression gives

$$[x_1, x_2]^T = (A^T A)^{-1} A^T b = [2.1212, 0.4545]^T.$$

The two regression lines are show in Figure 2.

In general, L_2 regression is more sensitive to outliers than L_1 regression, due to the squaring of the errors. However, L_2 regression is easier to compute.

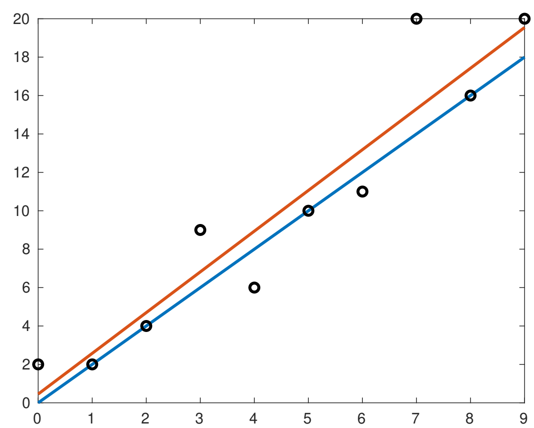

Figure 2: Blue: L_1 regression. Red: L_2 regression.