MAT3100, Spring 2020
Compulsory Assignment 2

Deadline 23 April, 14:30

Problem 1

1a)

Consider the linear programming problem

$$\max c^T x, \quad \text{subject to } Ax \le b,\ x \ge 0, \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $x, c \in \mathbb{R}^n$. State the weak and strong duality theorems. Moreover, prove the weak duality theorem.

1b)

Let $x = (x_1, x_2, \ldots, x_n)^T$ be primal feasible and $y = (y_1, y_2, \ldots, y_m)^T$ dual feasible. Denote by $(w_1, w_2, \ldots, w_m)$ the corresponding primal slack variables and $(z_1, z_2, \ldots, z_n)$ the corresponding dual slack variables. Suppose x is optimal for the primal problem and y is optimal for the dual problem. State and prove the complementary slackness equations.

1c)

Consider the linear programming problem

$$\begin{array}{lrcrcl} \text{maximize} & 3x_1 & + & 2x_2 & & \\ \text{subject to} & & & & & \\ & 2x_1 & + & x_2 & \le & 4, \\ & 2x_1 & + & 3x_2 & \le & 6, \\ & & & x_1, x_2 & \ge & 0. \end{array} \tag{2}$$

Show that $x^\star = (3/2, 1)$ is primal feasible and $y^\star = (5/4, 1/4)$ is dual feasible. Moreover, show that $x^\star$ is in fact an optimal solution of (2).

Problem 2

In order to heal heart diseases it is essential to detect them early. In this assignment you are going to analyze how an imaginary heart disease detection system works. First, suppose each heart cell has a certain "level of disease", which is some real number between 0 (perfectly healthy cell) and 1 (completely corrupted cell). The goal is to detect the level of disease of each cell in the heart, in order to assign the patient a localized treatment. Our detector essentially consists in an Ω-ray machine, which shoots rays through the heart and keeps track of the total disease that each ray encounters, by summing up the level of disease of every cell along the path of the ray. The weakness of this method is that a ray can give information about the total quantity of disease it meets, but not *in which point* of its way the disease is located. However, the Ω-ray machine can shoot many rays, so that we hope to ultimately be able to reconstruct—at least partially—the disease-map of the heart.

To simplify the analysis, let us put ourselves into the weird setting of a 2-dimensional heart, having a discrete rectangular shape. That is, we view the heart as an $m \times n$ rectangular grid. Each cell of the grid corresponds then to a cell in the heart, so that we can define a *disease matrix* $D = \{d_{ij}\} \in \mathbb{R}^{m,n}$ whose (i, j)-th entry is the level of disease of the (i, j)-th cell of the heart.

We want each entry of D to be a real number between 0 and 1; that is,

$$0 \leq d_{ij} \leq 1, \quad i = 1, \ldots, m, \; j = 1, \ldots, n.$$

Suppose we shoot an horizontal Ω-ray through each row of the heart, and a vertical Ω-ray through each column. Given $i \in \{1, \ldots, m\}$, the i-th horizontal ray will send back to the detector information corresponding to the total disease of the i-th row of D; i.e., the number

$$r_i = \sum_{j=1}^{n} d_{ij}.$$

Analogously, given $j \in \{1, \ldots, n\}$, the j-th vertical ray will send to the detector the number

$$c_j = \sum_{i=1}^{m} d_{ij}.$$

Clearly, the following equation must hold:

$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j.$$

The goal is now to reconstruct D from the known arrays $r = (r_1, \ldots, r_m)$ and $c = (c_1, \ldots, c_n)$. NOTE: The matrix D we want to find has mn entries, while we are only given $m+n$ known quantities. This means that the matrix D cannot be uniquely determined just by r and c. Therefore, the problem is to find *one* of possibly many configurations for D.

(a)

Consider the following LP problem:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} && (3) \\ \text{subject to} \quad & \\ & 0 \le x_{ij} \le 1, && i = 1, \ldots, m,\ j = 1, \ldots, n \\ & \sum_{j=1}^{n} x_{ij} \le r_i, && i = 1, \ldots, m \\ & \sum_{i=1}^{m} x_{ij} \le c_j, && j = 1, \ldots, n. \end{aligned}$$

Prove that the problem in the text has a solution if and only if (3) has an optimal solution $\hat{x} = \{\hat{x}_{ij}\}$ with corresponding objective function value s, where

$$s := \sum_{i=1}^{m} r_i = \sum_{j=1}^{n} c_j.$$

In this case the matrix $\hat{X} := \{\hat{x}_{ij}\}$ is a feasible disease matrix.

(b)

Suppose that

$$\begin{aligned} m &= 5, \\ n &= 7, \\ r &= (3.2, 2.0, 2.8, 5.7, 3.3), \\ c &= (4.0, 2.1, 2.2, 3.2, 1.9, 1.3, 2.3). \end{aligned}$$

Implement an algorithm to solve the LP problem (3) with the data given above. You can use the routine simplex.m (with pivot.m) on the course page or some other implementation of the simplex method. Does the solution you obtain correspond to a possible disease matrix? Explain why or why not.

Your delivery should be a short report summarizing your work as a **single pdf file**, submitted through canvas.