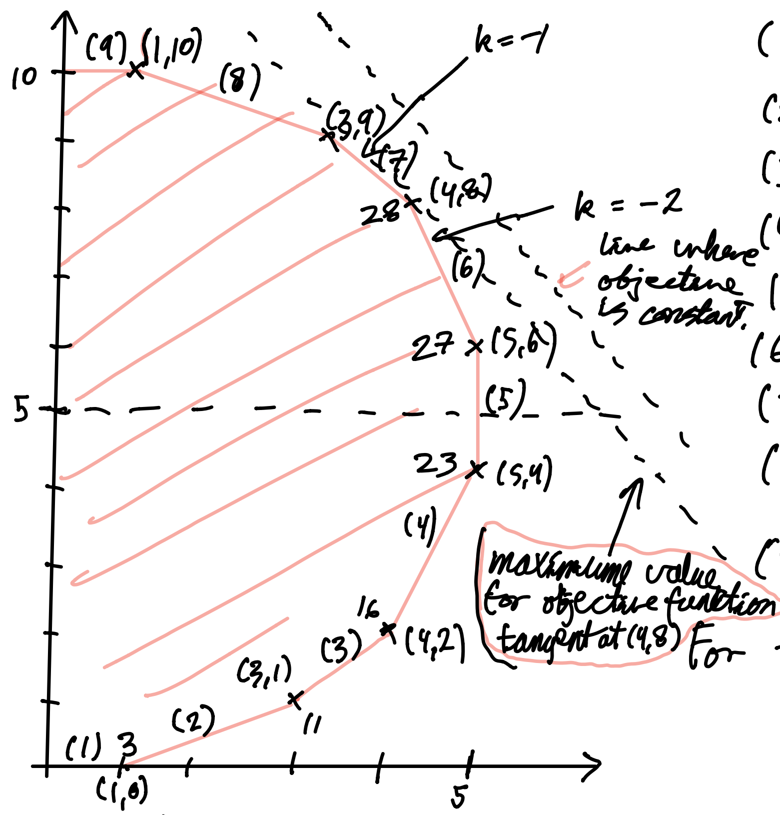
Exercise 2.16 (see exercise 2.8 also)



(1) $x_1, x_2 \geq 0$

(2) $x_1 - 2x_2 \leq 1$ $\quad w_1$

(3) $x_1 - x_2 \leq 2$ $\quad w_2$

(4) $2x_1 - x_2 \leq 6$ $\quad w_3$

(5) $x_1 \leq 5$ $\quad w_4$

(6) $2x_1 + x_2 \leq 16$ $\quad w_5$

(7) $x_1 + x_2 \leq 12$ $\quad w_6$

(8) $x_1 + 2x_2 \leq 21$ $\quad w_7$

(9) $x_2 \leq 10$ $\quad w_8$

For the lines, $k = \frac{1}{2}, 1, 2,$
$-2, -1, -\frac{1}{2}$

Each line segment corresponds to a slack being zero for one of the constraints.

objective : $3x_1 + 2x_2$

$$3x_1 + 2x_2 = k$$

$$x_2 = \underbrace{-\tfrac{3}{2}}_{\text{derivative } -\frac{3}{2}} x_1 + k.$$

See geometrically that $(x_1, x_2) = (4,8)$ gives max value : $3 \cdot 4 + 2 \cdot 8 = 12 + 16 = \underline{28}$

Simplex starts with $x_1 = x_2 = 0$.
maximum coefficient rule chooses x_1

Following the feasible region counterclockwise from the origin, the following variables are zero:

	nonbasic vars	obj. value
(1):	(x_1, x_2)	0
(2):	(x_2, w_1)	3
(3):	(w_1, w_2)	11
(4):	(w_2, w_3)	16
(5):	(w_3, w_4)	23
(6):	(w_4, w_5)	27
(7):	(w_5, w_6)	28

We see geometrically that the objective increases at each iteration.

If you run simplex with maximum coefficient rule, this order will be produced.

Simplex1 will yield a different order (hard to see, would be better to write solution x at all steps in the algorithm,

Exercise 2.18

$\xi = \cdots + c_j x_j$ becomes basic in the objective (entering)

$c_j > 0$

suppose x_k becomes nonbasic : $x_k = b_k - a_{kj} x_j - \cdots$

$$x_j = \frac{b_k}{a_{kj}} - \frac{1}{a_{kj}} x_k - \cdots$$

$$c_j x_j = c_j\left(\frac{b_k}{a_{kj}} - \frac{1}{a_{kj}} x_k - \cdots\right) = c_j \frac{b_k}{a_{kj}} - \underbrace{\frac{c_j}{a_{kj}}} x_k - \cdots$$

$c_j > 0$

$a_{kj} > 0$

so $-\frac{c_j}{a_{kj}} < 0$

Since the coefficient of x_k is <0 in the objective, it can't become basic in the next iteration.

Exercise 2.19

$$\max \quad \sum_{j=1}^{n} p_j x_j$$

$$\text{subj. to} \quad \sum_{j=1}^{n} q_j x_j \le \beta \qquad\qquad \sum_{j=1}^{n} p_j = \sum_{j=1}^{n} q_j = 1$$

$$0 \le x_j \le 1 \qquad\qquad p_i, q_j > 0$$

$$\text{Assume} \quad \frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n}.$$

β supposed to be small, positive

Easy to solve as follows without simplex:

Set $y_j = q_j x_j$, problem is tranformed to

$$\max \sum_{j=1}^{n} \frac{p_j}{q_j} y_j$$

$$\text{subj. to} \quad \sum_{j=1}^{n} y_j = \beta \qquad\qquad 0 \le y_j \le q_j$$

Clearly we must set $y_n = \min(\beta, q_n) = \beta$

$$\Rightarrow x_n = \frac{\beta}{q_n}, \quad \text{and} \quad \begin{array}{l} x_1 = \cdots = x_{n-1} = 0 \\ y_1 = \cdots = y_{n-1} = 0 \end{array}$$

optimal value: $\underline{\underline{\frac{p_n}{q_n}\beta}}$

With simplex:

	ratios
entering	
$\zeta = \qquad p_1 x_1 + \cdots\cdots + p_n x_n$	
$w_1 = 1 \quad - x_1$	
$\vdots$	
$w_n = 1 \qquad\qquad - x_n$	1
$w_{n+1} = \beta \quad - q_1 x_1 - \cdots \quad - q_n x_n$	$\frac{q_n}{\beta} > 1 \Rightarrow w_{n+1}$ leaving
$x_n = \frac{\beta}{q_n} - \frac{q_1}{q_n} x_1 - \cdots \quad - \frac{1}{q_n} w_{n+1}$	

$$\xi = \sum_{j=1}^{n-1} p_j x_j + p_n x_n$$

$$= \sum_{j=1}^{n-1} p_j x_j + \frac{p_n}{q_n}\left(\beta - \sum_{j=1}^{n-1} q_j x_j - w_{n+1}\right)$$

$$= \frac{p_n}{q_n}\beta + \sum_{j=1}^{n-1}\left(p_j - \frac{p_n}{q_n} q_j\right) x_j - \frac{p_n}{q_n} w_{n+1}$$

$$= \frac{p_n}{q_n}\beta + \sum_{j=1}^{n-1} q_j \underbrace{\left(\frac{p_j}{q_j} - \frac{p_n}{q_n}\right)}_{<0} x_j - \frac{p_n}{q_n} w_{n+1}$$

This is an optimal dictionary, optimal value is $\frac{p_n}{q_n}\beta$

also, $x_1 = \cdots = x_{n-1} = 0$, $x_n = \frac{\beta}{q_n}$

$$\Rightarrow \vec{x} = \left(0, \ldots, 0, \frac{\beta}{q_n}\right)$$

Exercise 3.1

$$\begin{array}{ll} \max & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subj. to} & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 \le 0 \\ & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 \le 0 \\ & x_1 \le 1 \end{array}$$

entering

$$\begin{array}{lrl} & \xi = & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{leavi} & w_1 = & \varepsilon_1 - 0.5x_1 + 5.5x_2 + 2.5x_3 - 9x_4 \\ & w_2 = & \varepsilon_2 - 0.5x_1 + 1.5x_2 + 0.5x_3 - x_4 \\ & w_3 = & 1 + \varepsilon_3 - x_1 \\ & x_1 = & 2\varepsilon_2 - 2w_2 + 3x_2 + x_3 - 2x_4 \end{array}$$

We obtain:

$$
\begin{aligned}
& & & & & & & & & \text{entering} & \\
\xi &= & & 20\varepsilon_2 & & -20w_2 & -27x_2 & +x_3 & -44x_4 \\
w_1 &= & \varepsilon_1 & -\varepsilon_2 & & +w_2 & +4x_2 & +2x_3 & -8x_4 \\
x_1 &= & & 2\varepsilon_2 & & -2w_2 & +3x_2 & +x_3 & -2x_4 \\
\text{leaving } w_3 &= 1 & & -2\varepsilon_2 & +\varepsilon_3 & +2w_2 & -3x_2 & -x_3 & +2x_4
\end{aligned}
$$

$$x_3 = 1 - 2\varepsilon_2 + \varepsilon_3 + 2w_2 - 3x_2 - w_3 + 2x_4$$

$$
\begin{aligned}
\xi &= 1 + 18\varepsilon_2 + \varepsilon_3 - 18w_2 - 30x_2 - w_3 - 42x_4 \\
w_1 &= 2 + \varepsilon_1 - 5\varepsilon_2 + 2\varepsilon_3 + 5w_2 - 2x_2 - 2w_3 - 4x_4 \\
x_1 &= 1 + \varepsilon_3 - w_3 \\
x_3 &= 1 - 2\varepsilon_2 + \varepsilon_3 + 2w_2 - 3x_2 - w_3 + 2x_4
\end{aligned}
$$

This dictionary is optimal! Delete the ε's:

$$
\begin{aligned}
\xi &= 1 - 18w_2 - 30x_2 - w_3 - 42x_4 \\
w_1 &= 2 + 5w_2 - 2x_2 - 2w_3 - 4x_4 \\
x_1 &= 1 - w_3 \\
x_3 &= 1 + 2w_2 - 3x_2 - w_3 + 2x_4
\end{aligned}
$$

$\Rightarrow$ optimal value is 1, obtained for $x_2 = x_4 = w_2 = w_3 = 0$

$w_1 = 2$, $x_1 = x_3 = 1$

$\Rightarrow \vec{x} = (1,0,1,0)$

Exercise 3.2 (same system using Blands rule)

$$
\begin{array}{lllllll}
 & & \text{entering} & & & & \text{ratios:} \\
\xi = & & 10x_1 & -57x_2 & -9x_3 & -24x_4 & \\
\text{leaving } w_1 = & & -0.5x_1 & +5.5x_2 & +2.5x_3 & -9x_4 & \infty \quad \text{Leaving (Blands)} \\
w_2 = & & -0.5x_1 & +1.5x_2 & +0.5x_3 & -x_4 & \infty \\
w_3 = & 1 & -x_1 & & & & \frac{1}{1}=1
\end{array}
$$

(pivot(1,1))

$$x_1 = -2w_1 + 11x_2 + 5x_3 - 18x_4$$

$$
\begin{array}{lllllll}
 & & & \text{entering} & & & \text{ratios:} \\
\xi = & & -20w_1 & +53x_2 & +41x_3 & -204x_4 & \\
x_1 = & & -2w_1 & +11x_2 & +5x_3 & -18x_4 & \frac{-11}{0} = -\infty \\
w_2 = & & w_1 & -4x_2 & -2x_3 & +8x_4 & \frac{4}{0} = \infty \Rightarrow \text{leaving} \\
w_3 = & 1 & +2w_1 & -11x_2 & -5x_3 & +18x_4 & \frac{11}{1} = 11
\end{array}
$$

(pivot(2,2))

$$
\begin{array}{lllllll}
 & & & & \text{entering} & & \text{ratios} \\
\xi = & & -\frac{27}{4}w_1 & -\frac{53}{4}w_2 & +\frac{29}{2}x_3 & -98x_4 & \\
x_1 = & & \frac{3}{4}w_1 & -\frac{11}{4}w_2 & -\frac{1}{2}x_3 & +4x_4 & \frac{\frac{1}{2}}{0} = \infty \quad x_1 \text{ leaving} \\
x_2 = & & \frac{1}{4}w_1 & -\frac{1}{4}w_2 & -\frac{1}{2}x_3 & +2x_4 & \frac{\frac{1}{2}}{0} = \infty \\
w_3 = & 1 & -\frac{3}{4}w_1 & \frac{11}{4}w_2 & +\frac{1}{2}x_3 & -4x_4 & \frac{-\frac{1}{2}}{1} = -\frac{1}{2}
\end{array}
$$

(pivot(3,1))

Use simplex for the remaining pivots.

Exercise 3.4

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{subj. to} \quad \sum_{j=1}^{n} a_{ij} x_j \le 0 \qquad i=1,\dots,m$$

$$x_j \ge 0 \qquad j=1,\dots,n$$

Assume that all $c_j \le 0$. Then this is optimal, and $\vec{x}=\vec{0}$ optimal solution.

Assume that $c_k > 0$ for some k. Set x_k as entering.

	ratios
$\xi = c_1x_1 + \dots + c_n x_n$	
$w_1 = -a_{11}x_1 - \dots - a_{1n}x_n$	$\frac{a_{1k}}{0}$
$\vdots$	
$w_m = -a_{m1}x_1 - \dots - a_{mn}x_n$	$\frac{a_{mk}}{0}$

If all ratios ≤ 0, then we can increase x_k to infinity $\Rightarrow$ unbounded.

Assume that some ratio is >0, for instance: $\frac{a_{ck}}{0} = \infty$

We must have that $a_{ck} > 0$.

Constraint c: $\quad 0 - a_{ck}x_k = 0 \quad \Rightarrow x_k = 0$

This shows that you can't increase the entering variable.

So, after all pivots, $\vec{x}$ remains at 0.

This completes the proof.