Chapter 5: Duality theory

To every LP-problem (denote it by (P)) there is another LP-problem (denoted (D)) called the dual problem.
(P) is also called the primal problem.
Turns out that the dual problem of (D) is (P).

Usefulness of duality:

- Can give us quick bounds for the optimal value of (P)
- One can solve (P) by solving (D)
  (D) may be more efficiently solved.

Standard form of (P)

$$\boxed{\begin{array}{lll} \max & \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \dots, m \\ & x_j \geq 0 & j = 1, \dots, n \end{array}}$$

Dual problem (D)

$$\boxed{\begin{array}{lll} \min & \sum_{i=1}^{m} b_i y_i & \\ \text{s.t.} & \sum_{i=1}^{m} y_i a_{ij} \geq c_j & j = 1, \dots, n \\ & y_i \geq 0 & i = 1, \dots, m \end{array}}$$

Note:
- For (P) rows of A correspond to constraints
  columns of A correspond to variables
- For (D) columns of A correspond to constraints
  rows of A correspond to variables
- b_i's are right hand sides in (P), but in the objective function in (D)
- c_j's are in the objective function in (P), but are right hand sides in (D)
- constraint inequalities are reversed.
- max is replaced by min.

Motivation result for duality

Theorem 5.1 (Weak duality)

Let $x_1, \dots, x_n$ be feasible for (P)

-11- $y_1, \dots, y_m$ ——— 11 ——— (D)

Then $\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i$

Proof: $\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \leq \sum_{i=1}^{m} b_i y_i$ ∎

Example:

(P) $\max\ 2x_1 + 2x_2$
s.t. $x_1 + 2x_2 \leq 1$
$2x_1 + x_2 \leq 1$
$x_1, x_2 \geq 0$

(D) $\min\ y_1 + y_2$
s.t. $y_1 + 2y_2 \geq 2$
$2y_1 + y_2 \geq 2$
$y_1, y_2 \geq 0$

We see that $(\frac{1}{4}, \frac{1}{4})$ is feasible for (P) objective value 1.

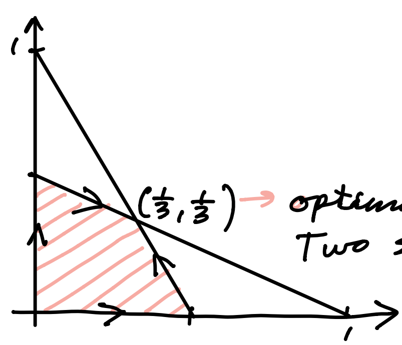
We see that $(1,1)$ is feasible for (D) objective value 2.

Therefore, optimal value for (P), and that for (D), both lie in $[1,2]$.

We also see that $(\frac{1}{3}, \frac{1}{3})$ is feasible for (P) objective value $\frac{4}{3}$.

We also see that $(\frac{2}{3}, \frac{2}{3})$ is feasible for (D) objective value $\frac{4}{3}$.

Therefore (P) and (D) both have $\frac{4}{3}$ as optimal value, due to weak duality.



→ optimal value

Two steps of simplex suffice

Interpretation of (D)

Assume $\vec{x}$ is feasible for (P), $\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i=1,\dots,m$

Multiply inequality i with $y_i \geqslant 0$, and add together:

$$\sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) \le \sum_{i=1}^{m} b_i y_i$$

Here, choose y_i so that $\underbrace{\sum_{i=1}^{m} y_i a_{ij} \geqslant c_j}$

combined coefficients $\geqslant$ those in the objective function.

$$\sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \le \sum_{i=1}^{m} b_i y_i$$

This means that $\sum_{i=1}^{m} y_i b_i$ (with $\sum_{i=1}^{m} y_i a_{ij} \geqslant c_j$) is an upper bound for (P).

Best possible upper bound for (P) is then

$$\boxed{\begin{aligned} \min \quad & \sum_{i=1}^{m} b_i y_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} y_i a_{ij} \geqslant c_j \quad j=1,\dots,n \end{aligned}}$$

which is the dual problem!

So the dual problem is: Find the best way to combine the constraints in (P), so that the combined coefficients dominate those in the objective function.

So, optimal value of (P) $\le$ optimal value of (D).

Can we have inequality?

Theorem 5.2 (Strong duality)

If (P) has an optimal solution $x^* = (x_1^*, \dots, x_n^*)$, then (D) also has an optimal solution $y^* = (y_1^*, \dots, y_m^*)$ so that
$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Proof: Idea is that the simplex method solves the primal and dual problems simultaneously.

(P)
$$\boxed{\begin{aligned} \max\ & \vec{c}^{\,T}\vec{x} \\ \text{s.t. } & A\vec{x} \le \vec{b} \\ & \vec{x} \ge 0 \end{aligned}}$$

(D) rewritten
$$\boxed{\begin{aligned} \max\ & -\vec{b}^{\,T}\vec{y} \\ \text{s.t. } & -A^T\vec{y} \le -\vec{c} \\ & \vec{y} \ge 0 \end{aligned}}$$

Dictionaries:

$$\left(\begin{array}{c|c} & \vec{c}^{\,T} \\ \hline \vec{b} & A \end{array}\right) \xrightarrow{\text{negative transpose}} \begin{pmatrix} & -\vec{b}^{\,T} \\ -\vec{c} & -A^T \end{pmatrix}$$

pivot. m with r leaving, and s entering:

$$D^{new} = D - \frac{1}{D_{r,s}} D_{:,s} D_{r,:}$$

$$D^{new}_{:,s} = -\frac{1}{D_{r,s}} D_{:,s}$$

$$D^{new}_{r,:} = \frac{1}{D_{r,s}} D_{r,:}$$

$$D_{r,s} = \frac{1}{D_{r,s}}$$

For the dual problem, let s be leaving, r entering:

pivot applied to (P)

$$\begin{array}{cc} & \begin{matrix} \phantom{d-\frac{bc}{a}} & s \end{matrix} \\ \begin{matrix} r \\ \end{matrix} & \begin{pmatrix} b & a \\ d & c \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{matrix} \phantom{d-\frac{bc}{a}} & s \end{matrix} \\ \begin{matrix} r \\ \end{matrix} & \begin{pmatrix} \frac{b}{a} & \frac{1}{a} \\ d-\frac{bc}{a} & -\frac{c}{a} \end{pmatrix} \end{array}$$

pivot applied to (D)

$$\begin{array}{cc} & \begin{matrix} & r \end{matrix} \\ \begin{matrix} \\ s \\ \end{matrix} & \begin{pmatrix} & -d \\ -c & -a \\ & -b \end{pmatrix} \end{array} \to \begin{array}{cc} & \begin{matrix} \phantom{\frac{c}{a}} & r \end{matrix} \\ \begin{matrix} \\ s \\ \end{matrix} & \begin{pmatrix} & -d+\frac{bc}{a} \\ \frac{c}{a} & -\frac{1}{a} \\ & -\frac{b}{a} \end{pmatrix} \end{array}$$

Still negative transposes of oneanother!

Lemma: The negative transpose property is preserved if (r,s)-pivot is applied to (P), and (s,r) pivot is applied to (D).

Example:

(P): entering

$$\begin{aligned} \eta &= \quad 4x_1 + x_2 + 3x_3 \\ w_1 &= 1 - x_1 - 4x_2 \\ \text{leaving } w_2 &= 3 - 3x_1 + x_2 - x_3 \end{aligned}$$

(D): entering

$$\begin{aligned} -\xi &= \quad -y_1 - 3y_2 \\ z_1 &= -4 + y_1 + 3y_2 \\ z_2 &= -1 + 4y_1 - y_2 \\ \text{leaving } z_3 &= -3 + y_2 \end{aligned}$$

neg. transpose

$$x_3 = 3 - 3x_1 + x_2 - w_2 \qquad\qquad y_2 = 3 + z_3$$

entering

$$\begin{aligned} \eta &= 9 - 5x_1 + 4x_2 - 3w_2 \\ \text{leaving } w_1 &= 1 - x_1 - 4x_2 \\ x_3 &= 3 - 3x_1 + x_2 - w_2 \end{aligned}$$

entering

$$\begin{aligned} -\xi &= -9 - y_1 - 3z_3 \\ z_1 &= 5 + y_1 + 3z_3 \\ \text{leaving } z_2 &= -4 + 4y_1 - z_3 \\ y_2 &= 3 + z_3 \end{aligned}$$

neg. transpose

$$x_2 = \tfrac{1}{4} - \tfrac{1}{4}x_1 - \tfrac{1}{4}w_1 \qquad\qquad y_1 = 1 + \tfrac{1}{4}z_2 + \tfrac{1}{4}z_3$$

$$\begin{aligned} \eta &= 10 - 6x_1 - w_1 - 3w_2 \\ x_2 &= \tfrac{1}{4} - \tfrac{1}{4}x_1 - \tfrac{1}{4}w_1 \\ x_3 &= \tfrac{13}{4} - \tfrac{13}{4}x_1 - \tfrac{1}{4}w_1 - w_2 \end{aligned}$$

$$\begin{aligned} -\xi &= -10 - \tfrac{1}{4}z_2 - \tfrac{13}{4}z_3 \\ z_1 &= 6 + \tfrac{1}{4}z_2 + \tfrac{13}{4}z_3 \\ y_1 &= 1 + \tfrac{1}{4}z_2 + \tfrac{1}{4}z_3 \\ y_2 &= 3 \phantom{+\tfrac{1}{4}z_2} + z_3 \end{aligned}$$

neg. transpose

After this pivot, both dictionaries are feasible and optimal!

In general:
- (P) is optimal $\Rightarrow$ (D) is feasible
- (P) is feasible $\Rightarrow$ (D) is optimal
- (P) is optimal and feasible $\Leftrightarrow$ (D) is optimal and feasible.

Proof for strong duality:

It is enough to find a dual feasible solution y^* so that

$$(*) \sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* .$$

Apply simplex to (P) to produce an optimal solution x^*

Final dictionary: $\xi = \xi^* + \sum_{j \in \mathcal{N}} \bar{c}_j x_j$

split variables into slack variables and non-slack variables.

defined here among the $\bar{c}_j$

$$\xi = \xi^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=1}^{m} d_i^* w_i$$

$(*1)$: $\xi^* = \sum_{j=1}^{n} c_j x_j^*$ (first and final dictionary compared).

Define $y_i^* = -d_i^*$

$(*2)$: $y_i^* \geq 0$ (final dictionary is optimal $\Rightarrow d_i^* \leq 0$)
(required for dual feasibility)

We also have:

$$\sum_{j=1}^{n} c_j x_j = \xi^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=1}^{m} d_i^* w_i$$

$$= \xi^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=1}^{m} (-y_i^*)\left(\underline{b_i} - \sum_{j=1}^{n} a_{ij} x_j\right)$$

$$= \underbrace{\xi^* - \sum_{i=1}^{m} b_i y_i^*}_{0} + \sum_{j=1}^{n} \underbrace{\left(c_j^* + \sum_{i=1}^{m} y_i^* a_{ij}\right)}_{c_j} x_j$$

Compare with LS:

Therefore: $(*3)$: $\xi^* = \sum_{i=1}^{m} b_i y_i^*$

$(*4)$: $c_j = c_j^* + \sum_{i=1}^{m} y_i^* a_{ij}$

$(*1)$ and $(*3)$ together imply $(*)$

Also, since $\hat{c}_j \le 0$ (optimality), $(*4)$ implies that

$(*5)$ $c_j \le \sum_{i=1}^{m} y_i^* a_{ij}$

$(*2)$, $(*5)$ imply dual feasibility ■

Complementary slack

Assume $\vec{x}$ is feasible for (P), $\vec{y}$ feasible for (D)

What is required for $\vec{x}$ to be optimal for (P),

and $\vec{y}$ —— " —— (D)?

Rewrite the proof for weak duality:

$$(*6): \sum_{j=1}^{n} c_j x_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij}\right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i \le \sum_{i=1}^{m} b_i y_i$$

If $\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$ ($\vec{x}$ optimal for (P), $\vec{y}$ for (D)), then the two inequalities must be equalities:

if $x_j > 0 \Rightarrow \sum_{i=1}^{m} y_i a_{ij} = c_j \quad j=1,\dots,n \Rightarrow \underbrace{z_j := \sum_{i=1}^{m} y_i a_{ij} - c_j}_{\text{dual slack variable}} = 0 \quad j=1,\dots,n$

if $y_i > 0 \Rightarrow \sum_{j=1}^{n} a_{ij} x_j = b_i \Rightarrow w_i = 0 \quad i=1,\dots,m$

Thus $x_j z_j = 0$ for all j, $y_i w_i = 0$ for all i

This is what we call complementary slack

Theorem 5.3 (Complementary slack)

Assume
- $\vec{x}$ feasible for (P)
- $\vec{y}$ feasible for (D)
- $w_1, \dots, w_m$ are primal slack variables
- $z_1, \dots, z_n$ are dual slack variables

Then $\vec{x}$ is optimal for (P) and $\vec{y}$ is optimal for (D) if and only if

$$\left.\begin{array}{ll} x_j z_j = 0 & j = 1, \dots, n \\ y_i w_i = 0 & i = 1, \dots, m \end{array}\right\} \text{complementary slack.}$$

Proof: $\Rightarrow$: Already shown

$\Leftarrow$: We have either $\left\{\begin{array}{l} x_j = 0 \Rightarrow \text{equality for term } j \text{ in first inequality in } (*6) \\ z_j = 0 \Rightarrow \text{equality for term } j \text{ in first inequality in } (*6) \end{array}\right.$

Since this holds for all j, the first inequality is an equality.
Similarly for the second inequality $\Rightarrow$ Both are optimal. ∎