Theorem 5.3 ensured that we can solve an LP problem be securing:

1) primal feasibility

2) dual feasibity

3) complementary slack

Many algorithms iterate so that two of these three are satisfied at all steps, and secure so that the third one is eventually satisfied.

- primal simplex:
  - 1) holds in all steps
  - 2) holds at the end
  - 3) holds in all steps (when $x_r \leftrightarrow x_s$ their dual variables also swap, so products zero
- primal-dual algorithms: 1) and 2) hold at all steps, and we iterate to secure 3)
- dual simplex: 2) and 3) hold at all steps, iterate to secure that 1) holds (primal feasibility).

  Attractive if there are more rows than columns in the dictionary (smaller number of pivots)

  Since dual dictionary is negative transpose of the primal dictionary, we never need to write it down.

Example:

$$\begin{aligned} \eta &= 12 - 4x_1 - x_2 - x_3 \\ w_1 &= -4 + 3x_1 - 11x_2 + x_3 \\ w_2 &= 3 - x_1 + 3x_2 - 2x_3 \end{aligned}$$

dual feasible, but not primal feasible.

Dual dictionary:

$$\begin{array}{lrl} & & \text{entering} \\ & -\xi = & -12 + 4y_1 - 3y_2 \\ & z_1 = & 4 - 3y_1 + y_2 \\ & z_2 = & 1 + 11y_1 - 3y_2 \\ \text{leaving} & z_3 = & 1 - y_1 + 2y_2 \end{array}$$

ratios: $\frac{3}{4}$, -11, 1 biggest, so z_3 leaves

seen as follows in the primal dictionary:

$$\begin{array}{lrl} & & \text{entering} \\ & \eta = & 12 - 4x_1 - x_2 - x_3 \\ \text{leaving} & w_1 = & -4 + 3x_1 - 11x_2 + x_3 \\ & w_2 = & 3 - x_1 + 3x_2 - 2x_3 \end{array}$$

in first column, find a negative number $\Rightarrow w_1$ is leaving

ratios: $\frac{3}{4}, -11, 1$
$\Rightarrow x_3$ entering.

$$x_3 = 4 - 3x_1 + 11x_2 + w_1$$

After first pivot:

$$\begin{array}{lrl} & \eta = & 8 - x_1 - 12x_2 - w_1 \\ & x_3 = & 4 - 3x_1 + 11x_2 + w_1 \\ \text{leaving} & w_2 = & -5 + 5x_1 - 19x_2 - 2w_2 \end{array}$$

ratios: 5, $-\frac{19}{12}$, -2
$\uparrow$
biggest $\Rightarrow x_1$ is entering

$$x_1 = 1 + \tfrac{1}{5}w_2 + \tfrac{19}{5}x_2 + \tfrac{2}{5}w_1$$

$$\begin{aligned} \eta &= 7 - \tfrac{1}{5}w_2 - 15.8x_2 - 1.4w_1 \\ x_3 &= 1 - \tfrac{3}{5}w_2 - \tfrac{2}{5}x_2 - \tfrac{1}{5}w_1 \\ x_1 &= 1 + \tfrac{1}{5}w_2 + \tfrac{19}{5}x_2 + \tfrac{2}{5}w_1 \end{aligned}$$

This is also primal feasible, so optimal!

$x_2 = 0$, $x_1 = x_3 = 1$.

General strategy:

1) replace objective function so that dictionary is dual feasible.
2) apply dual simplex method
3) Replace with original objective function (substitution)

Other LP forms

$$\boxed{\begin{array}{ll} \max & \vec{c}^T\vec{x} \\ \text{subj. to} & A\vec{x} = \vec{b} \\ & \vec{x} \geq \vec{0} \end{array}} \iff \boxed{\begin{array}{ll} \max & \vec{c}^T\vec{x} \\ \text{subj. to} & A\vec{x} \leq \vec{b} \\ & A\vec{x} \geq \vec{b} \\ & \vec{x} \geq 0 \end{array}} \iff \boxed{\begin{array}{ll} \max & \vec{c}^T\vec{x} \\ \text{subj. to} & A\vec{x} \leq \vec{b} \\ & -A\vec{x} \leq -\vec{b} \\ & \vec{x} \geq \vec{0} \end{array}}$$

$$\iff \boxed{\begin{array}{ll} \max & \vec{c}^T\vec{x} \\ \text{subj. to} & \begin{bmatrix} A \\ -A \end{bmatrix}\vec{x} \leq \begin{bmatrix} \vec{b} \\ -\vec{b} \end{bmatrix} \\ & \vec{x} \geq 0 \end{array}}$$

Dual problem

write $\vec{y} = [\vec{y}^+ \ \vec{y}^-]$

$$\boxed{\begin{array}{ll} \min & \begin{bmatrix} \vec{b} \\ -\vec{b} \end{bmatrix}^T \vec{y} \\ \text{s.t.} & [A^T \ -A^T]\vec{y} \geq \vec{c} \\ & \vec{y} \geq \vec{0} \end{array}} \iff \boxed{\begin{array}{ll} \min & \vec{b}^T\vec{y}_+ - \vec{b}^T\vec{y}_- \\ \text{s.t.} & A^T\vec{y}_+ - A^T\vec{y}_- \geq \vec{c} \\ & \vec{y}_+, \vec{y}_- \geq 0 \end{array}}$$

$$\iff \boxed{\begin{array}{ll} \min & \vec{b}^T(\vec{y}_+ - \vec{y}_-) \\ \text{s.t.} & A^T(\vec{y}_+ - \vec{y}_-) \geq \vec{c} \\ & \vec{y}_+, \vec{y}_- \geq 0 \end{array}} \quad \vec{y} = \vec{y}_+ - \vec{y}_- \iff \boxed{\begin{array}{ll} \min & \vec{b}^T\vec{y} \\ \text{s.t.} & A^T\vec{y} \geq \vec{c} \end{array}}$$

no $\vec{y} \geq \vec{0}$ constraint!

In other words: Equality constraints in (P) $\Rightarrow$ unconstrained variables in (D)

Similarly one can show:

Unconstrained variables in (P) $\Rightarrow$ equality constraints in (D)

Interpretation in terms of Lagrangian duality

Define $\Pi(x_1,\dots,x_n, y_1,\dots,y_m) = \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m}\sum_{j=1}^{n} y_i a_{ij} x_j + \sum_{i=1}^{m} y_i b_i$

This is called the Lagrangian function.

Rewrite Π in two ways:

$$\Pi(x_1,\dots,x_n, y_1,\dots,y_m) = \begin{cases} \sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} y_i a_{ij}\right) x_j + \sum_{i=1}^{m} y_i b_i \\[2ex] \sum_{i=1}^{m} \left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right) y_i + \sum_{j=1}^{n} c_j x_j \end{cases}$$

x fixed: $$\min_{y \geq 0} \Pi(x,y) = \begin{cases} -\infty & \text{if } b_i - \sum a_{ij} x_j < 0 \text{ for some } i \\ \sum_{j=1}^{n} c_j x_j & \text{if } b_i - \sum a_{ij} x_j \geq 0 \text{ all } i \\ & \text{(i.e., } \vec{x} \text{ feasible in (P))} \end{cases}$$

y fixed: $$\max_{x \geq 0} \Pi(x,y) = \begin{cases} \infty & \text{if } c_j - \sum y_i a_{ij} > 0 \text{ for some } j \\ \sum_{i=1}^{m} y_i b_i & \text{if } c_j - \sum y_i a_{ij} \leq 0 \\ & \text{(i.e. } \vec{y} \text{ feasible in (D))} \end{cases}$$

Therefore:

$$\max_{\vec{x} \geq 0} \min_{\vec{y} \geq 0} \Pi(x,y) = \max_{\vec{x} \text{ feasible in (P)}} \sum_{j=1}^{n} c_j x_j$$ i.e., the primal problem,

$$\min_{\vec{y} \geq 0} \max_{\vec{x} \geq 0} \Pi(x,y) = \min_{\vec{y} \text{ feasible in (D)}} \sum_{i=1}^{m} y_i b_i$$ i.e., the dual problem.

From strong duality, those are equal if an optimal solution exists.
Saddle points interpretation of optimum.

Chapter 6 Simplex method in matrix notation.

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\text{s.t.} \quad \sum a_{ij} x_j \le b_i \qquad i = 1, \dots, m$$

$$x_j \ge 0 \qquad j = 1, \dots, n$$

Define slack variables $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i$$

$$\underbrace{\begin{bmatrix} A' & I_m \end{bmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{\vec{b}}$$

Problem can be rewritten as

$$\max \ \vec{c}^T \vec{x} \qquad \text{where } \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\text{s.t.} \quad A\vec{x} = \vec{b}$$

$$\vec{x} \ge \vec{0}$$

Let B be the basis we have after simplex.

Set $\vec{x} = \begin{pmatrix} \vec{x}_B \\ \vec{x}_N \end{pmatrix}$ (place basic variables first)

Set $A = [B \ N]$ (reorganize columns of A so that columns that correspond to basic variables come first)

$Ax = [B \ N] \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix} = B\vec{x}_B + N\vec{x}_N = \vec{b}$ can express x_B in terms of x_N if B is invertible

$$\vec{x}_B = B^{-1}\vec{b} - B^{-1}N x_N$$

Also $\vec{c} = \begin{bmatrix} \vec{c}_B \\ \vec{c}_N \end{bmatrix}$ in same way, so that

$$\vec{c}^T \vec{x} = \begin{bmatrix} \vec{c}_B \\ \vec{c}_N \end{bmatrix}^T \begin{bmatrix} \vec{x}_B \\ \vec{x}_N \end{bmatrix} = \vec{c}_B^T x_B + \vec{c}_N^T \vec{x}_N$$

objective: $\xi = \vec{c}_B^T \vec{x}_B + \vec{c}_N \vec{x}_N$

$$= \vec{c}_B^T (B^{-1}\vec{b} - B^{-1}N\vec{x}_N) + \vec{c}_N \vec{x}_N$$

$$= \vec{c}_B^T B^{-1}\vec{b} - ((B^{-1}N)^T c_B - c_N)^T x_N$$

Summary:

new notation

$$(*P) \quad \boxed{\begin{aligned} \xi &= \vec{c}_B^T B^{-1}\vec{b} - ((B^{-1}N)^T c_B - c_N)^T x_N \\ x_B &= B^{-1}\vec{b} - B^{-1}N\vec{x}_N \end{aligned}}$$

old notation

$$\xi = \bar{\xi} + \sum \bar{c}_j \bar{x}_j$$

$$= \vec{b} - \vec{A}\vec{x}_N$$

Compare new and old notation:

$$\vec{c}_B^T B^{-1}\vec{b} = \bar{\xi}$$

$$c_N - (B^{-1}N)^T c_B = \vec{\bar{c}}$$

$$B^{-1}\vec{b} = \vec{\bar{b}}$$

$$B^{-1}N = \vec{\bar{A}}'$$

Basic solution:

$$\vec{x}_B^* = B^{-1}\vec{b}$$

$$\vec{x}_N^* = \vec{0}$$

Matrix form of the dual problem

Since dual slack variables are complementary to primal variables, we renumber variables as follows:

$$(z_1, \ldots, z_n, y_1, \ldots, y_m) \to (z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m})$$

complementary to:

$$(x_1, \ldots, x_n, w_1, \ldots, w_m) \to (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$$

Dual dictionary (take negative transpose of $(*P)$

$$(*D) \quad \boxed{\begin{aligned} -\eta &= -c_B^T B^{-1}\vec{b} - (B^{-1}\vec{b})^T \vec{z}_B \\ z_N &= (B^{-1}N)^T \vec{c}_B - \vec{c}_N + (B^{-1}N)^T \vec{z}_B \end{aligned}}$$

Basic solution

$$z_B^* = \vec{0}$$

$$z_N^* = (B^{-1}N)^T \vec{c}_B - \vec{c}_N$$

To simplify (*P) and (*D): Set $\xi^* = c_B^T B^{-1} b$, and use expressions for $x_B^*, \underset{0}{\underset{\parallel}{x_N^*}}, \underset{0}{\underset{\parallel}{z_B^*}}, z_N^*$

(*P) $\boxed{\begin{aligned} \xi &= \xi^* - (z_N^*)^T x_N \\ x_B &= x_B^* - B^{-1}N\, x_N \end{aligned}}$ (*D) $\boxed{\begin{aligned} -\eta &= -\xi^* - (x_B^*)^T z_B \\ z_N &= z_N^* + (B^{-1}N)^T \vec{z}_B \end{aligned}}$

Alternative description of simplex:

- partition the $n+m$ variables into m basic variables and n nonbasic variables so that B is invertible.
- Compute x_B^* and z_N^* (with $x_N^*=0$, $z_B^*=0$) as above.
- Assume $x_B^* \geq 0$ (primal feasibility)

Step 1 if $z_N^* \geq 0$ stop (dual feasibility $\Rightarrow$ optimality)

Step 2 select entering variable j so that $z_j^* \leq 0$
j is leaving in the dual problem.

Step 3 Compute primal step direction $\Delta \vec{x}_B$:
increase x_j, keep others at 0, i.e., $\vec{x}_N = t \cdot \vec{e}_j$, $t \geq 0$.

$$\vec{x}_B = \vec{x}_B^* - t\,\underbrace{(B^{-1}Ne_j)}_{:=\Delta x_B}$$

step 4: Compute primal step length: Largest t so that $\vec{x}_B \geq \vec{0}$

i.e.,
$$\vec{x}_B^* - t\,\Delta x_B \geq 0$$
$$\vec{x}_B^* \geq t\,\Delta \vec{x}_B$$
$$\frac{1}{t} \geq \frac{\Delta x_i}{x_i^*} \quad \text{for all } i \in B$$

i.e., choose t so that $\frac{1}{t} = \max_{i \in B} \frac{\Delta x_i}{x_i^*} \Rightarrow t = \frac{1}{\max_{i\in B} \frac{\Delta x_i}{x_i^*}}$

step 5 Select leaving variable: Any $i \in B$ that achieves the maximum in step 4. (i is entering in the dual problem)

step 6 Compute dual step direction Δz_N

$$\Delta z_N = -(B^{-1}N)^T \vec{e}_i$$

Step 7 Compute dual step length. j is leaving in (D)

$$0 = z_j = z_j^* - s\Delta z_j \quad \Rightarrow s = \frac{z_j^*}{\Delta z_j}$$

step 8 Update current primal and dual solutions:

$$x_j^* = t \qquad\qquad z_i^* = s$$

$$x_B^* := x_B^* - t\Delta x_B \qquad\qquad z_N^* := z_N^* - s\Delta z_N$$

step 9 Update basis:

in B: replace i with j

in N: replace j with i

Similar deductions apply for dual simplex, see algorithm on page 102.

Example: see section 6.3.