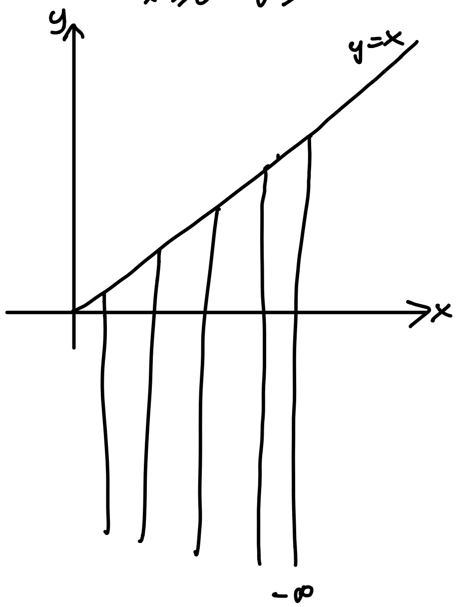
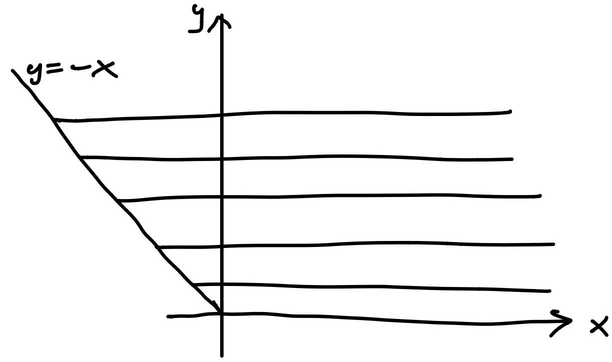
5.12

$\max_{x \geq 0} \min_{y \geq 0} (x-y)$



$= \max_{x \geq 0} -\infty = -\infty$

$\min_{y \geq 0} \max_{x \geq 0} (x-y)$



$= \min_{y \geq 0} \infty = \infty$

so minmax / maxmin give different values.

This is the Lagrangian function with $A = 0$,
$\vec{c} = (1,0)$
$\vec{b} = (0,-1)$

---

primal: $\max \ x$
s.t. $0 \leq 0$
$0 \leq -1$
infeasible

dual $\min \ -y$
s.t.
$0 \geq 1$
$0 \geq 0$
infeasible

5.15 (P)

n variables
n+1 constraints

(D)

$$\max \; p^T x$$
$$\text{s.t.} \begin{bmatrix} q^T \\ I \end{bmatrix} \vec{x} \le \begin{bmatrix} \beta \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\min \begin{bmatrix} \beta \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}$$
$$\text{s.t. } [q \; I]\vec{y} \ge \vec{p}$$

$$\begin{aligned}
\xi &= 0 + p_1 x_1 + \cdots + p_n x_n \\
w_0 &= \beta - q_1 x_1 - \cdots - q_n x_n \\
w_1 &= 1 - x_1 \\
&\vdots \\
w_n &= 1 - x_n
\end{aligned}$$

$$\begin{aligned}
-\eta &= 0 - \beta y_0 - y_1 - \cdots - y_n \\
z_1 &= -p_1 + q_1 y_0 + y_1 \\
&\vdots \\
z_n &= -p_n + q_n y_0 + y_n
\end{aligned}$$

Will show that the stated $\vec{x}, \vec{y}$ satisfy complementary slack.
Will not use simplex.
Stated expression:

$$\vec{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{\beta - \sum_{j=k+1}^{n} q_j}{q_k} \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{matrix} \}\, k-1 \\ \\ \}\, n-k \end{matrix}$$

$$\begin{aligned}
w_0 &= \beta - \sum q_j x_j \\
&= \beta - q_j \frac{\beta - \sum_{k+1}^{n} q_j}{q_j} - \sum_{k+1}^{n} q_j \\
&= \beta - \beta + \sum_{k+1}^{n} q_j - \sum_{k+1}^{n} q_j = 0
\end{aligned}$$

$$w_k = 1 - x_k = 1 - \frac{\beta - \sum_{k+1}^{n} q_j}{q_k} = -\frac{\beta - \sum_{k}^{n} q_j}{q_k}$$

$$\Rightarrow \vec{w} = \begin{pmatrix} 0 \\ \vdots \\ w_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \}\, k-1$$

$$\vec{y} = \begin{pmatrix} \dfrac{p_k}{q_k} \\ 0 \\ \vdots \\ 0 \\ q_j\left(\dfrac{p_j}{q_j} - \dfrac{p_k}{q_k}\right) \end{pmatrix} \begin{matrix} \}\, k \\ \}\, j \ge k \end{matrix} \ge \vec{0}$$

It is clear that $\vec{y}\,\vec{w} = \vec{0}$

z_j: $\quad j \le k$: $\quad = -p_j + q_j y_0 + y_j$

$$= -p_j + q_j \frac{p_k}{q_k} = q_j\left(\frac{p_k}{q_k} - \frac{p_j}{q_j}\right) \ge 0$$

z_j: $\quad j > k$: $\quad = -p_j + q_j y_0 + y_j$

$$= -p_j + q_j \frac{p_k}{q_k} + q_j\left(\frac{p_j}{q_j} - \frac{p_k}{q_k}\right)$$

$$= -p_j + q_j \frac{p_k}{q_k} + p_j - q_j \frac{p_k}{q_k} = 0$$

$$\vec{z} = \left(\begin{array}{c} q_j\left(\frac{p_k}{q_k} - \frac{p_j}{q_j}\right) \\ \\ 0 \\ \end{array}\right) \begin{array}{l} \}\ k \text{ entries} \\ \\ \}\ n-k \text{ entries} \end{array} \ge \vec{0}$$

It is clear that $\vec{x} \cdot \vec{z} = \vec{0}$

Primal feasibility:

By definition of k: $\quad \sum_{k+1}^{n} q_j \le \beta \quad , \quad \sum_{k}^{n} q_j > \beta$

$$x_k = \frac{\beta - \sum_{k+1}^{n} q_j}{q_k} \ge 0$$

$$w_k = -\frac{\beta - \sum_{k}^{n} q_j}{q_k} > 0$$

This shows that $\vec{x}, \vec{w} \ge \vec{0} \Rightarrow$ primal feasibility.

Dual feasibility: $\quad \vec{y} \ge \vec{0}$ is clear, $\vec{z} \ge \vec{0}$ is also clear

$\Rightarrow$ dual feasibility.

6.1

$$\begin{aligned} \max \quad & -6x_1 + 32x_2 - 9x_3 \\ \text{s.t.} \quad & -2x_1 + 10x_2 - 3x_3 \le -6 \\ & x_1 - 7x_2 + 2x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

$$\begin{aligned} \xi &= 18 - 3x_4 + 2x_2 \\ x_3 &= 2 - x_4 + 4x_2 - 2x_5 \\ x_1 &= \quad\; 2x_4 - x_2 + 3x_5 \end{aligned}$$

a) Basic variables: x_3, x_1 $\quad B = \{3, 1\}$

Nonbasic variables: x_4, x_2, x_5 $\quad N = \{4, 2, 5\}$

b) $x_B^* = (x_3, x_1) = (2, 0)$

c) $z_N^* = (y_4, y_2, y_5) = (3, -2, 0)$

d) $-B^{-1}N$ can be found in dictionary $\begin{pmatrix} -1 & 4 & -2 \\ 2 & -1 & 3 \end{pmatrix}$

$$A = \begin{pmatrix} -2 & 10 & -3 & 1 & 0 \\ 1 & -7 & 2 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix} \quad N = \begin{pmatrix} 1 & 10 & 0 \\ 0 & -7 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} \quad B^{-1}N = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 1 & -3 \end{pmatrix}$$

e) feasible $x_B^* \ge 0$

f) not optimal since not $z_N^* \ge 0$

g) dictionary is degenerate, due to the second constraint.

pivot: x_2 enters, x_1 leaves

6.6

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & a \leq Ax \leq b \\ & l \leq x \leq u \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & -Ax \leq -a \\ & Ax \leq b \\ & -x \leq -l \\ & x \leq u \end{aligned}$$

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & \begin{bmatrix} -A \\ A \\ -I \\ I \end{bmatrix} \vec{x} \leq \begin{bmatrix} -a \\ b \\ -l \\ u \end{bmatrix} \end{aligned}$$

dual problem:

$$\begin{aligned} \min \quad & c^T y \\ \text{s.t.} \quad & \begin{bmatrix} -A^T & A^T & -I & I \end{bmatrix} y \geq c \end{aligned}$$

7.1

$$\begin{aligned} \max \quad & x_1 + 2x_2 + x_3 + x_4 \\ \text{s.t.} \quad & 2x_1 + x_2 + 5x_3 + x_4 \leq 8 \\ & 2x_1 + 2x_2 + 4x_4 \leq 12 \\ & 3x_1 + x_2 + 2x_3 \leq 18 \end{aligned}$$

$$\begin{aligned} \xi &= 12.4 - 1.2x_1 - 0.2x_5 - 0.9x_6 - 2.8x_4 \\ x_2 &= 6 - x_1 - 0.5x_6 - 2x_4 \\ x_3 &= 0.4 - 0.2x_1 - 0.2x_5 + 0.1x_6 + 0.2x_4 \\ x_7 &= 11.2 - 1.6x_1 + 0.4x_5 + 0.3x_6 + 1.6x_4 \end{aligned}$$

a) objective changes from $c = (1,2,1,1,\underbrace{0,0,0}_{\text{slack}})$ to $(3,2,1,1,\underbrace{0,0,0}_{\text{slack}})$

here $B = \{2,3,7\}$ $\quad \Delta\vec{c} = (2,0,0,0,0,0,0)$

$N = \{1,5,6,4\}$ $\quad \Delta\vec{c}_B = (0,0,0)$

$\Delta\vec{c}_N = (2,0,0,0)$

coefficient matrix in dictionary is $-B^{-1}N = \begin{pmatrix} -1 & 0 & -0.5 & -2 \\ -0.2 & -0.2 & 0.1 & 0.2 \\ -1.6 & 0.4 & 0.3 & 1.6 \end{pmatrix}$

when c changes, z_N^* changes, but x_B^* (primal feasibility preserved)

$z_N^* = (1.2, 0.2, 0.9, 2.8)$

(7.1): $\Delta z_N = (B^{-1}N)^T \underbrace{\Delta c_B}_{0} - \Delta c_N = (-2,0,0,0)$

$z_N = z_N^* + \Delta z_N = \begin{pmatrix} -0.8 \\ 0.2 \\ 0.9 \\ 2.8 \end{pmatrix}$ ← not dual feasible

objective value changes as $\xi^* = c_B^T B^{-1} b$, i.e it changes with $\underbrace{\Delta c_B^T}_{0} B^{-1} b = 0$, so objective value does not change.

New dictionary:

						ratios
$\xi =$	12.4	$+0.8x_1$	$-0.2x_5$	$-0.9x_6$	$-2.8x_4$	
$x_2 =$	6	$-x_1$		$-0.5x_6$	$-2x_4$	$\frac{1}{6}$
$x_3 =$	0.4	$-0.2x_1$	$-0.2x_5$	$+0.1x_6$	$+0.2x_4$	$\frac{1}{2}$, x_3 leaves
$x_7 =$	11.2	$-1.6x_1$	$+0.4x_5$	$+0.3x_6$	$+1.6x_4$	$\frac{1.6}{11.2}$

$x_1 = 2 - 5x_3 - x_5 + 0.5x_6 + x_4$

$\xi = 14 - 4x_3 - x_5 - 0.5x_6 - 2x_4$

$x_2 = 4 + \ldots$

$x_1 = 2 - 5x_3 - x_5 + 0.5x_6 + x_4$

$x_7 = 8 + \ldots$

optimal! $\xi^* = 14$, $\vec{x} = (\underbrace{2,4,0,0}, 0,0,8)$

b) objective changes from $c=(1,2,1,1,0,0,0)$ to $(1,2,0.5,1,0,0,0)$

from before:

$B = \{2,3,7\}$

$N = \{1,5,6,4\}$

$z_N^* = \underline{(1.2, 0.2, 0.9, 2.8)}$

$\Delta \vec{c} = (0,0,-0.5,0,0,0,0)$

$\Delta \vec{c}_B = (0,-0.5,0)$

$\Delta \vec{c}_N = (0,0,0,0)$

$$\Delta z_N = \underbrace{(B^{-1}N)^T}_{\text{as before}} \Delta c_B - \underbrace{\Delta c_N}_{0} = \begin{pmatrix} 1 & 0.2 & 1.6 \\ 0 & 0.2 & -0.4 \\ 0.5 & -0.1 & -0.3 \\ 2 & -0.2 & -1.6 \end{pmatrix} \begin{pmatrix} 0 \\ -0.5 \\ 0 \end{pmatrix} = \underline{\begin{pmatrix} -0.1 \\ -0.1 \\ 0.05 \\ 0.1 \end{pmatrix}}$$

$$z_N = z_N^* + \Delta z_N = \begin{pmatrix} 1.1 \\ 0.1 \\ 0.95 \\ 2.9 \end{pmatrix}$$ feasible, so dictionary is still optimal.

new optimal value:

$$B = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 0 & 0.5 & 0 \\ 0.2 & -0.1 & 0 \\ -0.4 & -0.3 & 1 \end{pmatrix}$$

$$\xi = \xi^* + \Delta c_B^T B^{-1} \vec{b}$$

$$= 12.4 + (0,-0.5,0) B^{-1} \begin{pmatrix} 8 \\ 12 \\ 18 \end{pmatrix}$$

$$= 12.4 + (-0.1, 0.05, 0) \begin{pmatrix} 8 \\ 12 \\ 18 \end{pmatrix} = 12.4 - 0.8 + 0.6 = \underline{12.2}$$

Same optimal solution $\vec{x}$.

c) $\vec{b}$ changes from $(8,12,18)$ to $(8,26,18) \Rightarrow \Delta\vec{b} = (0,14,0)$

now z_N^* does not change (dual feasibility secured.

From dictionary: $x_B^* = (6, 0.4, 11.2)$. This will change when $\vec{b}$ changes.

$$\Delta x_B = B^{-1} \Delta \vec{b}$$ (follows from $x_B^* = B^{-1}\vec{b}$)

$$= \begin{pmatrix} 0 & 0.5 & 0 \\ 0.2 & -0.1 & 0 \\ -0.4 & -0.3 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 14 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -1.4 \\ -4.2 \end{pmatrix}$$

$$x_B = x_B^* + \Delta x_B = \begin{pmatrix} 6 \\ 0.4 \\ 11.2 \end{pmatrix} + \begin{pmatrix} 7 \\ -1.4 \\ -4.2 \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \\ 7 \end{pmatrix}$$ not primal feasible!

updated objective value: $\xi = \xi^* + C_B^T B^{-1} \Delta b$

$$= 12.4 + (2,1,0)\begin{pmatrix}7\\-1.4\\-4.2\end{pmatrix} = 12.4 + 14 - 1.4 = 25$$

Update the (feasible) dual dictionary:

$$\begin{aligned}
-\eta &= -25 - 13y_2 + y_3 + 7y_7 \\
y_1 &= 1.2 + y_2 + 0.2y_3 + 1.6y_7 \\
y_5 &= 0.2 + 0.2y_3 - 0.4y_7 \\
y_6 &= 0.9 + 0.5y_2 - 0.1y_3 - 0.3y_7 \\
y_4 &= 2.8 + 2y_2 - 0.2y_3 - 1.6y_7
\end{aligned}$$

ratios

$-\frac{0.2}{1.2} < 0$

$-\frac{0.2}{0.2} < 0$

$\frac{0.1}{0.9} = \frac{1}{9}$, y_6 leaves

$\frac{0.2}{2.8} = \frac{1}{14}$

$$y_3 = 9 + 5y_2 - 10y_6 - 3y_7$$

$$\begin{aligned}
-\eta &= -16 - 8y_2 - 10y_6 - 10y_7 \\
y_1 &= 3 + \cdots \\
y_5 &= 2 + \cdots \\
y_3 &= 9 + 5y_2 - 10y_6 - 3y_7 \\
y_4 &= 1 + \cdots
\end{aligned}$$

optimal!

primal dictionary is

$$\begin{aligned}
\xi &= 16 - \cdots \\
x_2 &= 8 + \cdots \\
x_6 &= 10 + \cdots \\
x_7 &= 10 + \cdots
\end{aligned}$$

$\xi = 16$ is optimal value, $\vec{x} = (0, 8, 0, 0, 0, 10, 10)$