Exercises chap. 17

17.1

(P)

$$\boxed{\begin{array}{lrl} \max & -x_1 + x_2 & \\ \text{s.t.} & x_2 & \le 1 \\ & -x_1 & \le -1 \\ & x_1, x_2 \ge 0 & \end{array}}$$

(D)

$$\begin{array}{lrl} \min & y_1 - y_2 & \\ \text{s.t.} & -y_2 & \ge -1 \\ & y_1 & \ge 1 \\ & y_1, y_2 \ge 0 & \end{array}$$

$\Downarrow$

$$\boxed{\begin{array}{lrl} \max & -y_1 + y_2 & \\ & y_2 & \le 1 \\ & -y_1 & \le -1 \\ & y_1, y_2 \ge 0 & \end{array}}$$

See that (P) and (D) coincide)

If follows that $x_i = y_i$

$w_i = z_i$

Solution to barrier problem:

same $\left\{ \begin{array}{l} Ax + w = b \\ A^T y - z = c \end{array} \right.$

$\underbrace{x_j z_j = w_i y_i}_{\text{the same}} = \mu$

$\Longrightarrow$ $Ax + w = b$, $x_j w_j = \mu$ $\Rightarrow$ $\boxed{\begin{array}{rl} x_2 + w_1 &= 1 \\ -x_1 + w_2 &= -1 \\ x_1 w_1 &= \mu \\ x_2 w_2 &= \mu \end{array}}$

substitute $w_1 = 1 - x_2$, $w_2 = -1 + x_1$ from first two equations:

$$x_1 w_1 = x_1(1 - x_2) = x_2 w_2 = x_2(-1 + x_1) = \mu$$

$$x_1 - x_1 x_2 = -x_2 + x_1 x_2 = \mu \quad \Rightarrow \quad x_2 = \frac{x_1 - \mu}{x_1}$$

$$x_2(x_1 - 1) = \frac{x_1 - \mu}{x_1}(x_1 - 1) = \mu$$

$$(x_1 - \mu)(x_1 - 1) = \mu x_1$$

$$x_1^2 - (2\mu + 1)x_1 + \mu = 0$$

$$x_1 = \frac{2\mu + 1 \pm \sqrt{(2\mu+1)^2 - 4\mu}}{2} = \frac{2\mu + 1 \pm \sqrt{4\mu^2 + 1}}{2}$$

since $x_1 \ge 1$: $x_1 = \dfrac{2\mu + 1 + \sqrt{4\mu^2 + 1}}{2}$

$$x_2 = 1 - \frac{\mu}{x_1} = 1 - \frac{2\mu}{2\mu + 1 + \sqrt{4\mu^2+1}} = 1 - \frac{2\mu\left(2\mu + 1 - \sqrt{4\mu^2+1}\right)}{4\mu}$$

mult with $2\mu + 1 - \sqrt{4\mu^2+1}$ up and down

$$= 1 - \frac{2\mu + 1 - \sqrt{4\mu^2+1}}{2}$$

$$= \frac{1 - 2\mu + \sqrt{4\mu^2+1}}{2}$$

expressions for z_j, w_j are obtained by taking inverses: $x_j z_j = \mu$

As $\mu \to 0$: $x_1 \to 1$, $x_2 \to 1$

$w_1 \to 0$ $\quad w_2 \to 0$

Exercise 17.2

$$\max \quad (\cos\theta)x_1 + (\sin\theta)x_2$$
$$\text{s.t.} \quad x_1 \le 1$$
$$x_2 \le 1$$
$$x_1, x_2 \ge 0$$

Clear that optimal value is $\cos\theta + \sin\theta$ for $x_1 = x_2 = 1$ (for $0 \le \theta \le \frac{\pi}{2}$)

$Ax + w = b$:
$A^T y - z = c$

$$\begin{aligned} x_1 \quad\quad & + w_1 = 1 \\ x_2 \quad & + w_2 = 1 \\ y_1 \quad\quad & - z_1 = \cos\theta \\ y_2 \quad & - z_2 = \sin\theta \end{aligned}$$

1, 2

$$x_i z_i = y_i w_i = \mu$$

Split equations into two categories:

1-indices

1. $x_1 + w_1 = 1$
2. $y_1 - z_1 = \cos\theta$

$x_1 z_1 = y_1 w_1 = \mu$

$z_1 = \frac{\mu}{x_1} \qquad y_1 = \frac{\mu}{w_1}$

2-indices

$x_2 + w_2 = 1$
$y_2 - z_2 = \sin\theta$
$x_2 z_2 = y_2 w_2 = \mu$

1. $x_1 + w_1 = 1 \Rightarrow w_1 = 1 - x_1$
2. $\frac{\mu}{w_1} - \frac{\mu}{x_1} = \cos\theta$

$\frac{\mu}{1-x_1} - \frac{\mu}{x_1} = \cos\theta$

$\mu x_1 - \mu(1-x_1) = (x_1 - x_1^2)\cos\theta$, set $\lambda = \frac{\cos\theta}{\mu} = \frac{c}{\mu}$

$$2x_1 - 1 = \lambda x_1 - \lambda x_1^2$$
$$\lambda x_1^2 + (2-\lambda)x_1 - 1 = 0$$
$$x_1 = \frac{\lambda - 2 + \sqrt{(2-\lambda)^2 + 4\lambda}}{2\lambda} = \frac{\lambda - 2 \pm \sqrt{\lambda^2+4}}{2\lambda}$$

- can be discarded, since $x_1 \ge 0$

it follows that $x_1 = \frac{\lambda - 2 + \sqrt{\lambda^2+4}}{2\lambda} = \frac{\frac{c}{\mu} - 2 + \sqrt{\frac{c^2}{\mu^2}+4}}{2\frac{c}{\mu}} = \frac{c - 2\mu + \sqrt{c^2 + 4\mu^2}}{2c}$

similarly x_2 has same expression with $c = \sin\theta$.

$$\Rightarrow (x_1, x_2) = \left(\frac{\cos\theta - 2\mu + \sqrt{\cos^2\theta + 4\mu^2}}{2\cos\theta}, \frac{\sin\theta - 2\mu + \sqrt{\sin^2\theta + 4\mu^2}}{2\sin\theta} \right)$$

The other variables: $\vec{w} = 1 - \vec{x}$, $\vec{z} = \frac{\mu}{\vec{x}}$ $\quad \vec{y} = \frac{\mu}{\vec{w}}$.

$\mu \to 0$: $(x_1, x_2) \to (1,1)$

$\mu \to \infty$: $\sqrt{\cos^2\theta + 4\mu^2} - 2\mu = \frac{\cos^2\theta}{\sqrt{\cos^2\theta + 4\mu^2} + 2\mu} \to 0$

$$(x_1, x_2) \to \left(\frac{\cos\theta}{2\cos\theta}, \frac{\sin\theta}{2\sin\theta} \right) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right).$$

Exercise 17.3

$$L(x,w,y) = \underbrace{c^T x + \sum_j r_j \log x_j + \sum_i s_i \log w_i}_{f(x,w)} + y^T(b - Ax - w)$$

$$\frac{\partial L}{\partial x_j} = c_j + \frac{r_j}{x_j} - \sum y_i a_{ij} = 0 \Rightarrow -A^T \vec{y} + \vec{c} + X^{-1} r = 0$$

$$\frac{\partial L}{\partial w_i} = \frac{s_i}{w_i} - y_i = 0 \Rightarrow W^{-1} s - y = 0$$

$$\frac{\partial L}{\partial y_i} = b_i - \sum_j a_{ij} x_j - w_i = 0 \Rightarrow b - Ax - w = 0$$

Can be rewritten as

$$A^T y - \underbrace{X^{-1} r}_{z} = c$$

$$y = W^{-1} s$$

$$Ax + w = b$$

$$Ax + w = b$$

$$A^T y - z = c$$

$$y = W^{-1} s \Leftrightarrow Wy = s \Leftrightarrow WYe = s$$

$$z = X^{-1} r \Leftrightarrow Xz = r \Leftrightarrow XZe = r$$

$$\frac{\partial^2 f}{\partial x_j} = -\frac{r_j}{x_j^2} < 0$$

$$\frac{\partial^2 f}{\partial w^2} = -\frac{s_i}{w_i^2} < 0$$

All mixed derivatives vanish

$\Rightarrow H^2 f$ is negative definite

Theorem 17.1 Critical point is unique if it exists

Corollary 17.3: Critical point exists since primal feasible set is assumed to be bounded.

Exercise 17.4

Basic feasible solutions to (17.8): Obtained by setting all but one variable to zero. $\leftarrow V$

First order optimality conditions for (17.9):

$$\nabla f = \lambda \nabla g$$

$$\begin{pmatrix} 2c_1\xi_1 \\ \vdots \\ 2c_n\xi_n \end{pmatrix} = \lambda \begin{pmatrix} 2a_1\xi_1 \\ \vdots \\ 2a_n\xi_n \end{pmatrix} \quad \leftarrow W$$

Clearly, if all ξ_i except one is nonzero, this is fulfilled with $\lambda = \frac{c_i}{a_i}$, so $V \subset W$.
So, (17.9) will give us too many possibilities to consider, since we have to go through all basic feasible solutions.