Exercises chap. 11

11.3 a) Let $y \in \mathbb{R}^m$ be a randomized strategy for row player.

Want to find $\min\limits_{\substack{y \geq 0 \\ e^T y = 1}} \max\limits_{\substack{x \geq 0 \\ e^T x = 1}} y^T A x \to (x^*, y^*)$

s is as in the text.

$$\max_{\substack{x \geq 0 \\ e^T x = 1}} (y^T A x) = \max_j (y^T A)_j = \max_j \left(\sum_{i \neq r,s} y_i a_{ij} + y_r a_{rj} + y_s a_{sj} \right)$$

$$\overset{\text{row } r \geq \text{row } s}{\geq} \max_j \left(\sum_{i \neq r,s} y_i a_{ij} + a_{sj}(y_r + y_s) \right)$$

$$= \max_j \left(\sum_{i \neq r} y_i' a_{ij} \right)$$

where y' is y with entry r removed, and y_s is replaced with $y_r + y_s$. y' is a randomized strategy for the game where row r is removed ($A \to A_0$)

Let y^* be an optimal strategy for this new game. Then

$$\max_{x \geq 0} (y^*)^T A_0 x \leq \max \sum_{i \neq r} y_i' a_{ij} \overset{\text{proved above}}{\leq} \max_{\substack{x \geq 0 \\ e^T x = 1}} (y^T A x) \quad \text{for all } y$$

$$\Downarrow$$

$$\max_{x \geq 0} (y^*)^T A_0 x \leq \min_{\substack{y \geq 0 \\ e^T y = 1}} \max_{\substack{x \geq 0 \\ e^T x = 1}} y^T A x$$

$(y_1^*, \dots, y_{r-1}^*, 0, y_{r+1}^*, \dots) \Rightarrow$ if you insert a zero at place r in y^*, you get an optimal strategy for the game.

b) Similar (see notes from last year)

c) In summary: if row r dominates ~~anoter~~ another row, set $q_r^* = 0$

if col s is dominated by another column, set $y_s^* = 0$

see that col 1 dominates col 4 $\Rightarrow$ can drop col. 4

$$\begin{pmatrix} -6 & 2 & -4 & -7 & -5 \\ 0 & 4 & -2 & -9 & -1 \\ -7 & 3 & -3 & -8 & -2 \\ 2 & -3 & 6 & 0 & 3 \end{pmatrix}$$

$\downarrow$

$$\begin{pmatrix} -6 & 2 & -4 & -5 \\ 0 & 4 & -2 & -1 \\ -7 & 3 & -3 & -2 \\ 2 & -3 & 6 & 3 \end{pmatrix}$$ see that row 2 $\geq$ row 1 $\Rightarrow$ can drop row 2

$\downarrow$

$$\begin{pmatrix} -6 & 2 & -4 & -5 \\ -7 & 3 & -3 & -2 \\ 2 & -3 & 6 & 3 \end{pmatrix}$$ see that col 3 $\geq$ col. 1 $\Rightarrow$ can drop col. 1

$\downarrow$

$$\begin{pmatrix} 2 & -4 & -5 \\ 3 & -3 & -2 \\ -3 & 6 & 3 \end{pmatrix}$$ see that row 2 $\geq$ row 1 $\Rightarrow$ can drop row 2

$\downarrow$

$$\begin{pmatrix} 2 & -4 & -5 \\ -3 & 6 & 3 \end{pmatrix}$$ see that col 2 $\geq$ col 3 $\Rightarrow$ can drop col. 3

$\downarrow$

$$\begin{pmatrix} 2 & -4 \\ -3 & 6 \end{pmatrix}$$

11.2 A picks i, B picks j A wins : $a_{ij} = -1$

B wins : $a_{ij} = 1$

row player

$$\begin{pmatrix} 0 & 1 & & & & -1 \\ -1 & 0 & 1 & & & \\ & -1 & 0 & \ddots & & \\ & & & \ddots & & \\ & 1 & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}$$

see that row $4, 5, 6, \dots, 100$ dominate row 1, can remove those

Left with:

$$\begin{pmatrix} 0 & 1 & -1 & -1 & -1 & \\ -1 & 0 & 1 & -1 & -1 & \cdots \\ 1 & -1 & 0 & 1 & -1 & \end{pmatrix}$$

col 1 dominates col $4, 5, 6, \dots, 100$, can remove those

Left

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

This is the payoff matrix for the papers scissors rock game, where $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ was an optimal strategy (see Chap. 11)

11.5 (*) rth pure strategy of row player: row player chooses $y = e_r$
(**) sth pure strategy of col player: col player chooses $x = e_s$.

(*) col. player chooses x by $\max_{\substack{x \geq 0 \\ e^T x = 1}} (e_r)^T A x = \max_{\substack{x \geq 0 \\ e^T x = 1}} (Ax)_r$

(**) row player chooses y by $\min_{\substack{y \geq 0 \\ e^T y = 1}} y^T A e_s = \min_{\substack{y \geq 0 \\ e^T y = 1}} (y^T A)_s$

So, when is $\underbrace{\max_{\substack{x \geq 0 \\ e^T x = 1}} (Ax)_r}_{\max_k a_{kr}} = \underbrace{\min_{\substack{y \geq 0 \\ e^T y = 1}} (y^T A)_s}_{\min_c a_{sc}}$

max in rth column = min in sth row

Since: $\max_k a_{kr} \geq a_{sr} \geq \min_c a_{sc}$

We must have that $a_{sr} = \max_k a_{kr} = \min_c a_{sc}$

So: a_{sr} must be max in col r, and min in row s.

11.6 We have shown that, with $f(x) = \min_{\substack{y \geq 0 \\ e^T y = 1}} y^T A x$

$g(y) = \max_{\substack{x \geq 0 \\ e^T x = 1}} y^T A x$,

$\max_{\substack{x \geq 0 \\ e^T x = 1}} f(x)$ and $\min_{\substack{y \geq 0 \\ e^T y = 1}} g(y)$ are dual problems.

By the minmax theorem, there exist x^*, y^* so that
$f(x^*) = g(y^*)$

Weak duality: $\max_{\substack{x \geq 0 \\ e^T x = 1}} f(x) = \min_{\substack{y \geq 0 \\ e^T y = 1}} g(y)$

This shows that $\max_{x \geq 0, e^T x = 1} \min_{y \geq 0\, e^T y = 1} y^T A x = \min_{y \geq 0\, e^T y = 1} \max_{x \geq 0\, e^T x = 1} y^T A x$.