Chapter 14, Network flow problems

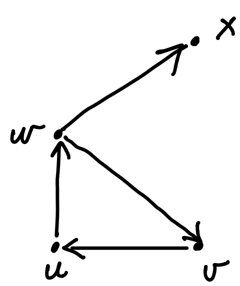
Mathematical models for flows in networks.

Models will be linear, and based on graphs

A directed graph is an ordered pair $\overbrace{(\mathcal{N}, \mathcal{A})}^{\text{network}}$ where

- $\mathcal{N}$ is a finite set, called nodes (vertices, or points)
- $\mathcal{A}$ is a finite set of ordered pairs of nodes, called edges (lines, or arcs)

Drawing a directed graph: (i,j) denotes the edge from node i to node j (use arrows)



$\left.\begin{array}{l} i: \text{initial node} \\ j: \text{terminal node} \end{array}\right\}$ end nodes of $e=(i,j)$

Here:

$\mathcal{N} = \{u, v, w, x\}$

$\mathcal{A} = \{(v,u), (u,w), (w,v), (w,x)\}$

A flow in a directed graph:

- Goods flow along the edges in the direction the edges show
- Each node has a certain supply or demand for goods, denoted b_i:
  a) $b_i > 0$: supply node
  b) $b_i < 0$: demand node
  c) $b_i = 0$: transshipment node

We will have that $\sum_{i \in \mathcal{N}} b_i = 0$: Everything stays within the graph: produced in graph $\Rightarrow$ consumed in graph.

x_{ij} denotes the flow in graph in edge (i,j)

x_{ij} often demanded to be nonnegative, and with an upper capacity : $0 \le x_{ij} \le \underbrace{a_{ij}}_{\text{not used for now}}$

With shipping along (i,j) there is associated a cost c_{ij} (the cost of shipping one unit). Total cost of shipping in graph is

$$\sum_{(i,j)\in A} c_{ij} x_{ij}$$

Flow balance equations:

$$\underbrace{\sum_{i:(i,k)\in A} x_{i,k}}_{\text{inflow to } k} - \underbrace{\sum_{j:(k,j)\in A} x_{k,j}}_{\text{outflow to } k} = -b_k$$

$b_k > 0$: supply node:
outflow > inflow
inflow − outflow < 0

multiply with −1 on both sides:

outflow − inflow = b_k (easier to remember).

Minimum cost network flow problem (MCF)

$$\boxed{\begin{aligned} \min \; & \sum_{(i,j)\in A} c_{ij} x_{ij} \\ \text{s.t.} \; & \sum_{(i,k)\in A} x_{ik} - \sum_{(k,j)\in A} x_{kj} = -b_k \\ & x \ge 0 \end{aligned}}$$

This is an LP problem with a special structure.

In matrix form:

$$(*) \quad \boxed{\begin{aligned} \min \; & c^T x \\ \text{s.t} \; & Ax = -b \\ & x \ge 0 \end{aligned}}$$

A : one row for each node ($|N|$ rows)
one column for each edge (i,j).
In column (i,j), entry j is 1, entry i is -1,
all other entries are zero.

in a row: a 1 for each incoming edge, and
a -1 for each outgoing edge.

A is also called a node-edge incidence matrix

Dual problem of $(*)$ is

$$(**) \quad \boxed{\begin{aligned} &\max \ -b^T y \\ &\text{s.t.} \quad A^T y + z = c \\ &\qquad\quad z \geq 0 \end{aligned}}$$

equality constraints in $(*)$
$\Downarrow$
y is unconstrained

y : one component for each node
z : one component for each edge

In component form:

$$\boxed{\begin{aligned} &\max \ -\sum b_i y_i \\ &\text{s.t} \quad y_j - y_i + z_{ij} = c_{ij} \quad (i,j) \in A \\ &z_{uv} \geq 0 \quad y_i \text{ unconstrained} \end{aligned}}$$

Equivalent to:

$$\boxed{\begin{aligned} &-\min \sum b_i y_i \\ &\text{s.t.} \quad y_j - y_i \leq c_{ij} \quad (i,j) \in A \end{aligned}}$$

This problem is feasible (set $y_i = 0$).

LP theory to find optimal solution to MCF:

complementary slack: $X_{uv} Z_{uv} = 0$ + feasibility $\begin{cases} \text{primal: } X \geq 0 \text{ flow balance} \\ \text{dual: } y_j - y_i \leq C_{ij} \text{ all } (i,j) \end{cases}$
(slacks in primal are 0)

$\Downarrow$

if $y_j < y_i + C_{ij}$ ($\Leftrightarrow Z_{ij} > 0$), then x_{ij} must be zero

MCF can be solved far more efficiently than general LP problems, due to their special structure.

Terms from graph theory

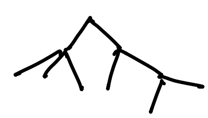
A path: a sequence $v_1, v_2, \dots, v_t$ of nodes where either (v_i, v_{i+1}) or (v_{i+1}, v_i) is in A for each i. v_1 and v_t are called end nodes of the path.

Connected graph: A graph where there is a path between any pair of nodes.

Cycle: A path where the end nodes coincide

A subgraph of (N, A): A network (N', A') where $N' \subseteq N$, $A' \subseteq A$, and all edges in A' have end nodes in N'.

A tree: A connected graph without cycles.
(the number of edges = number of nodes -1)



Spanning tree: A subgraph (N, A') without cycles.

We will explain that:

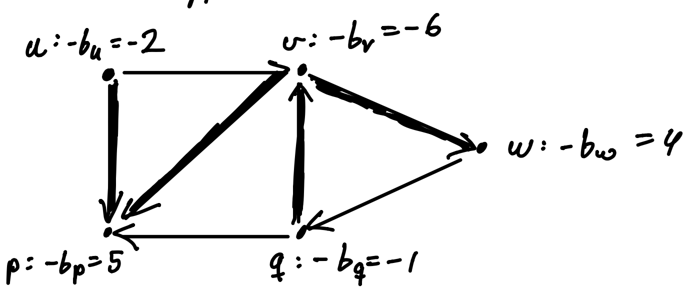
1) spanning trees correspond to LP bases
2) Calculation of basic solutions: performed by simple operations on spanning trees.
3) pivot $\to$ simple transformation of a spanning tree into another spanning tree.

For a given spanning tree we compute a so-called <u>tree solution</u> X as follows:

1/ Set $X_e = 0$ for all $e=(i,j)$ not in the spanning tree. (X_{ij} nonbasic)

2/ Starting at a <u>leaf node</u>, we can use the flow balance equation to compute each X_{ij}, where (i,j) is in the spanning tree (X_{ij} basic)

<u>Example</u>: spanning tree in bold
supplies/demands shown at the nodes.



1. $X_{uv} = X_{qp} = X_{wq} = 0$ (these are not in the spanning tree)

Four edge flows need to be computed. 3 leaf nodes:

2. u is leaf node: $\quad$ outflow − inflow $= b_u = 2$
$$\underline{X_{up} \qquad\qquad = 2}$$

3. w is leaf node: $\quad$ outflow − inflow $= b_w = -4$
$$-X_{vw} = -4$$
$$\underline{X_{vw} = 4}$$

4. q is leaf node: $\quad$ outflow − inflow $= b_q = 1$
$$\underline{X_{qv} \qquad\qquad = 1}$$

5. Flow balance in p: outflow − inflow = $b_p = -5$

$$-x_{up} - x_{vp} = -5$$
$$-2 - x_{vp} = -5$$
$$x_{vp} = 3$$

Alternatively:

Flow balance in v: outflow − inflow = $b_v = 6$

$$x_{vp} + x_{vw} - x_{qv} = 6$$
$$x_{vp} + 4 - 1 = 6$$
$$x_{vp} = 3$$

Algorithm for computing x given the spanning tree T:

1) Choose a leaf node in the spanning tree

2) Apply flow balance for that node to compute x_e for the unique edge in the spanning tree incident to that leaf.

3) Remove e (and leaf node) from T, and continue from 1) (x_e is needed in the other flow balance equations)

This procedure will always work, to find a tree solution.

Note: One flow balance equation is not used.

The equations we solved (variables are permuted)

$$
\begin{array}{lrrrrl}
2. & x_{up} & & & & = 2 \\
3. & & -x_{vw} & & & = -4 \\
4. & & & x_{qv} & & = 1 \\
5. & -x_{up} & & & -x_{vp} & = -5 \\
\text{alt. way:} & & x_{vw} & -x_{qv} & +x_{vp} & = 6
\end{array}
$$

See that this gives a 4×4 invertible submatrix of A (since the matrix is lower triangular)

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}$$

Each row: Flow balance for $|N|-1$ nodes (any one can be dropped)

Each column: edges in the spanning tree

Convince yourself: Any spanning tree gives rise to a $(|N|-1)\times(|N|-1)$ submatrix which is invertible. From this we can find x_e for e in the spanning tree. The others are set to 0.

If a graph is connected, we always have a spanning tree. A spanning tree will always give linearly independent rows as above, so rank $A \geq |N|-1$

Also rank$(A) < N$, since the sum of all rows is 0.

It follows that rank $A = |N|-1$

So: Given a spanning tree, the corresponding tree solution can be found as the basic solution where any node is $\underbrace{\text{taken out}}_{\text{called root node}}$ of the flow balance equations

Basic variables: Edges in the spanning tree

Nonbasic variables: Edges outside the spanning tree.

Removing root node, the problem looks as follows:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & \tilde{A} x = -\tilde{b} \\ & x \geq 0 \end{aligned}$$

A with one row removed

Theorem 14.1: A square submatrix of $\tilde{A}$ gives a basis if and only if the corresponding edges form a spanning tree.

Proof: From the above example: Any spanning tree gives a bases (since the matrix we end up with is triangular). Assume now that the columns do not form a spanning tree. Then we have $|N|-1$ edges that do not visit all $|N|$ nodes. Graph theory says that we then must have a cycle. Corresponding columns sum to 0, so we have linear dependence of the columns, so the matrix is not invertible $\Rightarrow$ no basis. ∎

So: spanning trees $\Leftrightarrow$ LP bases in $\tilde{A}$

How do we find the dual variables?

$$\max \; -\sum_{i \in N} b_i y_i$$

$$y_j - y_i + z_{ij} = c_{ij} \quad (i,j) \in A$$

Dual variable for the root node (y_r). is equivalent to having $y_r = 0$.

if (i,j) in spanning tree: x_{ij} is basic $\Rightarrow z_{ij}$ is nonbasic $\Rightarrow z_{ij} = 0$

$$y_j - y_i + z_{ij} = c_{ij}$$

$$y_j - y_i = c_{ij}$$

All y_i can be found from this (use that $y_r = 0$)

After the y_i have been computed, we can use to compute the z_{ij}.