Exercise 10

conv(S) is convex: Need to show that

$$(1-\lambda)\Big(\underbrace{\overset{x_j\in S}{\sum_j \lambda_j x_j}}_{\sum_j \lambda_j = 1}\Big) + \lambda\Big(\underbrace{\overset{y_k \in S}{\sum_k \mu_k y_k}}_{\sum \mu_k = 1}\Big) \in \operatorname{conv}(S)$$

$$= \sum_j (1-\lambda)\lambda_j x_j + \sum_k \lambda \mu_k y_k$$

The coefficients sum to $\sum_j (1-\lambda)\lambda_j + \sum_k \lambda\mu_k$

$$= (1-\lambda)\sum_j \lambda_j + \lambda \sum_k \mu_k = (1-\lambda)+\lambda = 1$$

so that conv(S) is convex.

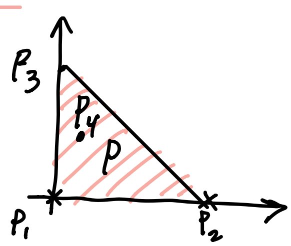
Note that convex hull consists of all such combinations:

$$\sum_j \lambda_j x_j = \lambda_1 x_1 + (1-\lambda_1)\sum_{j>1} \underbrace{\frac{\lambda_j}{1-\lambda_1}}_{} x_j$$

$$\sum_{j>1} \frac{\lambda_j}{1-\lambda_1} = \frac{\lambda_2+\dots+\lambda_n}{1-\lambda_1} = \frac{1-\lambda_1}{1-\lambda_1} = 1.$$

By induction, $\sum \lambda_j x_j$ is in the convex hull.

Exercise 11



Clearly $P = \operatorname{conv}\{\overbrace{P_1, P_2, P_3}^{S_0}\}$

$= \operatorname{conv}\{\underbrace{P_1, P_2, P_3, P_4}_{S}\}$

Exercise 12

Assume $S \subseteq T$.

Let $x_1, \dots, x_t \in S$

conv(S) : all points of the form $\sum \lambda_i x_i$, $x_i \in S$

conv(T) : all points of the form $\sum \lambda_i x_i$, $x_i \in T$

This shows that $\operatorname{conv}(S) \subseteq \operatorname{conv}(T)$

Exercise 13

We know that $S \subseteq \text{conv}(S)$

Need to show that $\underbrace{\text{conv}(S)}_{\substack{\sum \lambda_i x_i \in S \text{ if } S \text{ is convex} \\ \text{(shown above), exercise 10.}}} \subseteq S$

Exercise 14

Assume that $\|x\| < 1$, $x \neq 0$:

$$(1-\lambda)\underbrace{\frac{x}{\|x\|}}_{\in S} + \lambda\Big(\underbrace{-\frac{x}{\|x\|}}_{\in S}\Big) = (1-2\lambda)\frac{x}{\|x\|}$$

This equals x if $\frac{1-2\lambda}{\|x\|} = 1 \iff 1-2\lambda = \|x\|$

$$\Updownarrow$$

$$\lambda = \frac{1-\|x\|}{2}.$$

This shows that $x \in \text{conv}(S)$ whenever $\|x\| < 1$, $x \neq 0$. $\Rightarrow 0 < \lambda < \frac{1}{2}$

Clearly also $0 \in \text{conv}(S)$, since $0 = \frac{1}{2}\underbrace{(1,0)}_{\in S} + \frac{1}{2}\underbrace{(-1,0)}_{\in S}$

It follows that $S = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, since conv$(S)$ is the smallest convex set containing S, and $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ is convex.

Exercise 15

$S = \{(0,0), (1,0), (0,1)\}$

$$\text{conv}(S) = \lambda_1(0,0) + \lambda_2(1,0) + \lambda_3(0,0), \quad \lambda_1+\lambda_2+\lambda_3 = 1, \; 0 \leq \lambda_i$$

$$= (\lambda_2, \lambda_3) \quad \lambda_2 + \lambda_3 \leq 1 \quad \lambda_2, \lambda_3 \geq 0$$

$$= (x_2, x_3), \quad x_2 + x_3 \geq 0 \quad x_2, x_3 \geq 0$$

Exercise 16 $\quad S = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}$

Clearly, any convex combination of these points satisfy $0 \le x_i \le 1$

$\Rightarrow \operatorname{conv}(S) \subseteq \{x : 0 \le x_i \le 1\}$

convex combinations of $(0,0,0), (1,0,0), (0,1,0), (1,1,0)$ gives

$$\{(x,y,0) : 0 \le x,y \le 1\}$$

convex combinations of $(0,0,1), (1,0,1), (0,1,1), (1,1,1)$ gives

$$\{(x,y,1) : 0 \le x,y \le 1\}$$

Finally, $(x,y,z) = (1-z)(x,y,0) + z(x,y,1)$,

This shows that $\{x : 0 \le x_i \le 1\} \subseteq \operatorname{conv}(S)$.

It follows that $\{x : 0 \le x_i \le 1\} = \operatorname{conv}(S)$

$\operatorname{conv}(S \setminus \{1,1,1\})$:

Note that $x_1 + x_2 + x_3 \le 2$ for all remaining seven points

$\Rightarrow \quad x_1 + x_2 + x_3 \le 2$ for $\operatorname{conv}(S \setminus \{1,1,1\})$.

$\Rightarrow \operatorname{conv}(S \setminus \{1,1,1\}) \subseteq \{\underbrace{x : x_1 + x_2 + x_3 \le 2, 0 \le x_i \le 1}_{P}\}$

Assume that $(x_1, x_2, x_3) \in P \Rightarrow \underbrace{x_1 + x_2 + x_3 \le 2 \ , 0 \le x_i \le 1}$

Basic solutions for this feasible set are other seven points, which thus are the extreme points of P. Since P is bounded, so P is the convex hull of its extreme points (main theorem for polyhedra). So $P \subseteq \operatorname{conv}(S \setminus \{1,1,1\})$

It follows that $\underline{P = \mathrm{conv}(\ S \setminus \{1,1,1\})}$

Exercise 17

Lemma 4: $Ax \le b$ has solution $\Leftrightarrow y^T b \ge 0$ for all y, $y^T A = 0$, $y \ge 0$

$$\left.\begin{array}{l} Ax = b \\ x \ge 0 \end{array}\right\} \quad \left.\begin{array}{r} Ax \le b \\ -Ax \le -b \\ -x \le 0 \end{array}\right\} \quad \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} x \le \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}$$

This has a solution if

$$(y_1\ \ y_2\ \ y_3) \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \ge 0 \quad \text{for all } y, \ [y_1\ y_2\ y_3] \begin{bmatrix} A \\ -A \\ -I \end{bmatrix} = 0 \quad y \ge 0$$

$$\underbrace{(y_1 - y_2)}_{y \text{ unconstrained}}{}^T b \ge 0 \qquad \text{for all } y, \ (y_1 - y_2)^T A - y_3 = 0 \quad y \ge 0$$

$y_3 \ge 0$

$$\underline{y^T b \ge 0 \qquad \text{for all } y, \ y^T A \ge 0},$$

Exercise 18

Example 5:

(i) $P = \{ x \in \mathbb{R}^n : a_1 x_1 + \dots + a_n x_n = b \}$

extreme points: If two x_i's > 0: Then x cannot be extreme: since you can then increase one of these and decrease the other to obtain a decomposition of x ($\Rightarrow x$ not extreme):

$$\frac{1}{2}\Big(\sum_{i \ne r,s} a_i x_i + (a_r - \varepsilon_1) x_r + (a_s + \varepsilon_2) x_s\Big) \qquad \varepsilon_1 x_r = \varepsilon_2 x_s$$

$$+\frac{1}{2}\Big(\sum_{i \ne r,s} a_i x_i + (a_r + \varepsilon_1) x_r + (a_s - \varepsilon_2) x_s\Big) \qquad \varepsilon_1 x_r = \varepsilon_2 x_s$$

$a_r - \varepsilon_1,\ a_s - \varepsilon_2 > 0$

$$= \quad \sum_i a_i x_i = b$$

So: Only one x_i can be nonzero $\Rightarrow$ $x_i = \frac{b}{a_i}$,

which are clearly extreme.

(ii): Actually P is empty here.

(iii): Done in the text.