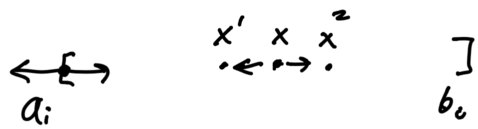
Exercise 19 $\quad P = \{ x \in \mathbb{R}^n : a_i \leq x_i \leq b_i \} \qquad a_i \leq b_i$

extreme points : $\quad x_i = \{ a_i , b_i \}$



$x = \frac{1}{2} x^1 + \frac{1}{2} x^2$

$x = \frac{1}{2} x^1 + \frac{1}{2} x^2$

Exercise 20 $\quad x_1 \geq 0 , \; x_2 \geq 0 \qquad x_1 + x_2 \leq 1 \qquad x_2 \leq \frac{1}{2}$

$$w_1 = 1 - x_1 - x_2$$
$$w_2 = \frac{1}{2} \qquad - x_2$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

possibilities for columns of B :

$\{1,2\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$

$B = \{1,2\}$, x_1, x_2 basic , $w_1 = w_2 = 0$

$\left.\begin{array}{r} x_1 + x_2 = 1 \\ x_2 = \frac{1}{2} \end{array}\right\} \Rightarrow x_1 = x_2 = \frac{1}{2} \qquad \vec{x} = (\frac{1}{2}, \frac{1}{2})$

feasible

$B = \{1,4\}$ $\quad x_1, w_2$ basic , $x_2 = w_1 = 0$

$w_2 = \frac{1}{2} - x_2 = \frac{1}{2} \geq 0 \qquad \begin{array}{r} x_1 + x_2 = 1 \\ x_2 = 0 \end{array} \Rightarrow \vec{x} = (1, 0)$

feasible

$B = \{2,3\}$ $\quad x_2, w_1$ basic , $x_1 = w_2 = 0 \Rightarrow x_2 = \frac{1}{2} \Rightarrow \vec{x} = (0, \frac{1}{2})$

$w_1 = 1 - x_1 - x_2 = \frac{1}{2} \geq 0$

feasible

$B = \{2,4\}$ $\quad x_2, w_2$ basic, $x_1 = w_1 = 0 \Rightarrow x_1 + x_2 = 1 \Rightarrow x_2 = 1$

$w_2 = \frac{1}{2} - x_2 = \frac{1}{2} - 1 = -\frac{1}{2} < 0 \qquad \Rightarrow \vec{x} = (0, 1)$

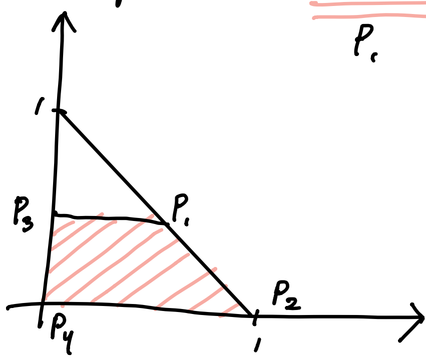
infeasible.

$B = \{3,4\}$ $\quad w_1, w_2$ basic $\Rightarrow x_1 = x_2 = 0 \Rightarrow \vec{x} = (0,0)$

$w_1 = 1, \; w_2 = \frac{1}{2}$ $\qquad$ feasible

---

4 feasible points : $\underbrace{(\tfrac{1}{2}, \tfrac{1}{2})}_{P_1}, \underbrace{(1,0)}_{P_2}, \underbrace{(0, \tfrac{1}{2})}_{P_3}, \underbrace{(0,0)}_{P_4}$



Exercise 21 $\quad x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0$

$$x_1 + x_2 + x_3 \leq 4$$
$$x_1 + x_2 \geq 1$$
$$x_3 \leq 2$$

$$\begin{aligned} w_1 &= 4 - x_1 - x_2 - x_3 \\ w_2 &= -1 + x_1 + x_2 \\ w_3 &= 2 \qquad\qquad\quad - x_3 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 + w_1 \qquad\qquad &= 4 \\ -x_1 - x_2 \qquad\quad + w_2 \quad &= -1 \\ x_3 \qquad\qquad + w_3 &= 2 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

possible choices : $\{1,3,4\}, \{1,3,5\}, \{1,3,6\}$
$\{1,4,6\}, \{1,5,6\},$
$\{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \ldots$

$B = \{1,3,4\}$: $\quad B x_B + N x_N = b$

$$x_B = B^{-1} b \overset{\text{matlab}}{=} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{matrix} x_1 \\ x_3 \\ w_1 \end{matrix} \qquad \begin{matrix} x_2 = 0 \\ w_2 = w_3 = 0 \end{matrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{feasible}$$

$B = \{1,3,5\}$ $\quad x_B = B^{-1}b = \begin{array}{c} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \end{array} \begin{array}{l} x_1 \\ x_3 \\ w_2 \end{array}$ $\quad x_2 = w_1 = w_3 = 0$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$ feasible

---

$B = \{1,3,6\}$ $\quad x_B = B^{-1}b = \begin{array}{c} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} \end{array} \begin{array}{l} x_1 \\ x_2 \\ w_3 \end{array}$ $\quad$ infeasible

---

$B = \{1,4,6\}$ $\quad x_B = B^{-1}b = \begin{array}{c} \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \end{array} \begin{array}{l} x_1 \\ w_1 \\ w_3 \end{array}$ $\quad x_2 = x_3 = w_2 = 0$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ feasible

---

Exercise 22 Fourier Motzkin on

$$x_1, x_2 \geq 0$$
$$x_1 + x_2 \leq 1$$
$$x_2 \leq \tfrac{1}{2}$$

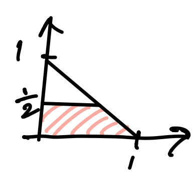
$$0 \leq x_1 \leq 1 - x_2 \qquad \begin{array}{l} x_2 \geq 0 \\ x_2 \leq \frac{1}{2} \end{array}$$

$0 \leq 1 - x_2 \Rightarrow x_2 \leq 1$, which is redundant since $x_2 \leq \frac{1}{2}$.

$$0 \leq x_2 \leq \tfrac{1}{2}$$

Feasible region: $\quad 0 \leq x_1 \leq 1 - x_2$

$\quad 0 \leq x_2 \leq \frac{1}{2}$



Exercise 23 Fourier Motzkin on

$$x_1, \underset{I_0}{x_2}, \underset{I_0}{x_3} \geq 0$$
$$x_1 + x_2 + x_3 \leq 4 \quad I_+$$
$$x_1 + x_2 \geq 1 \quad I_-$$
$$x_3 \leq 2 \quad I_0$$

$$\underbrace{\max(0, 1 - x_2)}_{I_-} \leq x_1 \leq \underbrace{4 - x_2 - x_3}_{I_+} \qquad \begin{array}{l} x_2, x_3 \geq 0 \\ x_3 \leq 2 \end{array}$$

$$\max(0, 1 - x_2) \leq 4 - x_2 - x_3$$
$$x_2, x_3 \geq 0$$
$$x_3 \leq 2$$

$\Rightarrow$ $0 \leq 4 - x_2 - x_3 \Rightarrow x_2 + x_3 \leq 4$

$1 - x_2 \leq 4 - x_2 - x_3 \Rightarrow x_3 \leq 3$

redundant.

We are left with:

$$0 \leq x_1 \leq 4 - x_2 - x_3$$

$$0 \leq 4 - x_2 - x_3$$
$$x_2, x_3 \geq 0$$
$$x_3 \leq 2$$
$\Rightarrow$

$2 \quad 0 \leq x_2 \leq 4 - x_3$

$3 \quad 0 \leq x_3 \leq 2$

$0 \leq 4 - x_3 \Leftrightarrow x_3 \leq 4$, redundant.

Exercise 24

$$a_{11}x_1 + a_{12}x_2 \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 \leq b_2$$
$$a_{31}x_1 + a_{32}x_2 \leq b_3$$
$$a_{41}x_1 + a_{42}x_2 \leq b_4$$

$i \in I_+ : \quad x_1 \leq \frac{1}{a_{i1}}(b_i - a_{i2}x_2)$

$i \in I_- \quad x_1 \geq \frac{1}{a_{i1}}(b_i - a_{i2}x_2)$

for $i \in I_+, j \in I_-$: $\frac{1}{a_{j1}}(b_j - a_{j2}x_2) \leq x_1 \leq \frac{1}{a_{i1}}(b_i - a_{i1}x_2)$