Exercise 14.3

a) primal flows: outflow − inflow $= b_i$ (flow balance at node i)

Flow balance at ⓕ: $-X_{af} = 0 \Rightarrow \underline{X_{af} = 0}$

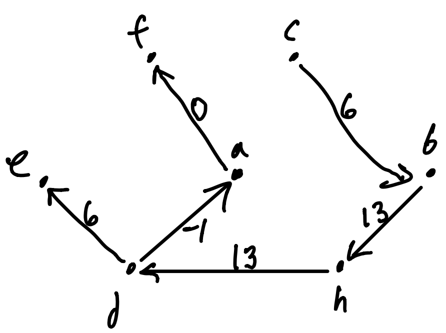
Flow balance at ⓐ: $X_{af} - X_{da} = 1 \Rightarrow \underline{X_{da} = -1}$

Flow balance at ⓔ: $-X_{de} = -6 \Rightarrow \underline{X_{de} = 6}$

Flow balance at ⓒ: $X_{cb} = 6 \Rightarrow \underline{X_{cb} = 6}$

Flow balance at ⓑ: $X_{bh} - X_{cb} = 7 \Rightarrow \underline{X_{bh} = 13}$

Flow balance at ⓗ: $X_{hd} - X_{bh} = 0 \Rightarrow \underline{X_{hd} = 13}$



b) Dual variables $y_j - y_i = C_{ij}$ for (i,j) in the spanning tree.

Let e be the root node, so that $y_e = 0$.

(d,e): $y_e - y_d = 6 \Rightarrow \underline{y_d = -6}$

(d,a): $y_a - y_d = 0 \Rightarrow \underline{y_a = -6}$

(a,f): $y_f - y_a = 6 \Rightarrow \underline{y_f = 0}$

(h,d): $y_d - y_h = -4 \Rightarrow y_h = y_d + 4 = \underline{-2}$

(b,h): $y_h - y_b = 14 \Rightarrow y_b = y_h - 14 = \underline{-16}$

(c,b): $y_b - y_c = 7 \Rightarrow y_c = y_b - 7 = -16 - 7 = \underline{-23}$

c) Dual slacks for nontree arcs (use that $y_j - y_i + z_{ij} = c_{ij}$

(e,f): $y_f - y_e + z_{ef} = c_{ef} \Rightarrow 0 - 0 + z_{ef} = 8 \Rightarrow z_{ef} = 8$

(c,f): $y_f - y_c + z_{cf} = c_{cf} \Rightarrow 0 - (-23) + z_{cf} = 3 \Rightarrow z_{cf} = -20$

(a,c): $y_c - y_a + z_{ac} = c_{ac} \Rightarrow -23 - (-6) + z_{ac} = 7 \Rightarrow z_{ac} = 24$

(a,e): $y_e - y_a + z_{ae} = c_{ae} \Rightarrow 0 - (-6) + z_{ae} = 3 \Rightarrow z_{ae} = -3$

(a,b): $y_b - y_a + z_{ab} = c_{ab} \Rightarrow -16 - (-6) + z_{ab} = 0 \Rightarrow z_{ab} = 10$

(h,a): $y_a - y_h + z_{ha} = c_{ha} \Rightarrow -6 - (-2) + z_{ha} = 1 \Rightarrow z_{ha} = 5$

Exercise 14.4 a) $\min \sum c_{ij} x_{ij} = -\max \sum -c_{ij} x_{ij}$

candidates: $-c_{ij} > 0 \Leftrightarrow c_{ij} < 0$

Largest coefficient rule: largest $-c_{ij}$ $\Updownarrow$ smallest c_{ij}

(we should pick the smallest negative c_{ij}

In the graph we see that $c_{ad} = -13$ is the smallest negative, so x_{ad} enters.

b) Leaving arc: Let x_{ad} be increased to ε

Adding (a,d) to the graph gives the cycle (a,d,b,e,g,a)

Flow balance equations within cycle

only flow in cycle can change when x_{ad} is increased

$$\begin{aligned}
\tilde{x}_{ad} &= \varepsilon \\
\tilde{x}_{ag} &= 1 - \varepsilon \Rightarrow \text{becomes zero first (for } \varepsilon = 1) \\
\tilde{x}_{ge} &= 14 - \varepsilon \\
\tilde{x}_{eb} &= 14 - \varepsilon \\
\tilde{x}_{bd} &= 6 - \varepsilon
\end{aligned}$$

so x_{ag} leaves

new values: $\tilde{x}_{ad} = 1$

$\tilde{x}_{ag} = 0$

$\tilde{x}_{ge} = 13$

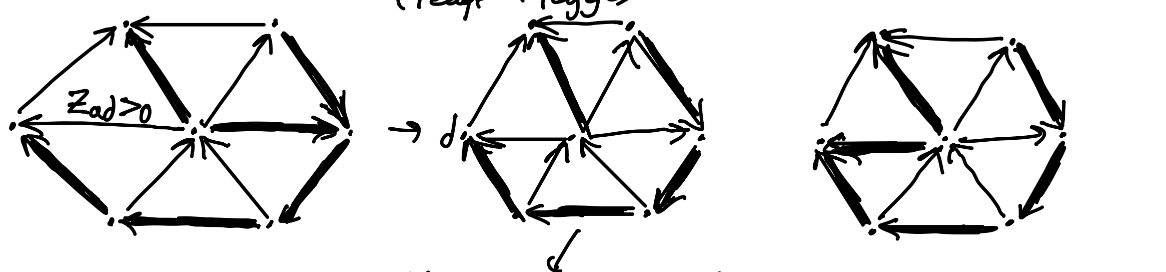
$\tilde{x}_{eb} = 13$

$\tilde{x}_{bd} = 5$

start

(a,g) taken out
two disjoint subtrees
(1 edge + 4 edges)

(a,d) added
new spanning tree



one subtre contains d, another does not

Assume that the root node is one of the nodes in the d-subtree

Since $y_j - y_i = c_{ij}$ in this subtree, none of the y_i change in this subtree (since y_i for the root does not change)

$$\left.\begin{array}{l} y_d - y_a + z_{ad} = c_{ad} \\ \quad\| \qquad \sim \qquad\qquad \| \\ \tilde{y}_d - \tilde{y}_a \qquad\quad = c_{ad} \end{array}\right\} \quad \begin{array}{l} \tilde{y}_a = y_a - z_{ad} \Rightarrow y_a \text{ decreased with } z_{ad} = -13 \\ \tilde{y}_a = y_a + 13 \end{array}$$

since $y_j - y_i = c_{ij}$ on the subtree containing a, all other y_i in this subtree are also increased with 13
(The y_i are not stated, however)

Since $y_j - y_i + z_{ij} = c_{ij}$ between nodes in the same subtree, z_{ij} is not changed when (i,j) has end nodes in the same subtree.

Arcs (i,j) bridging the two subtrees change z_{ij} as follows:

(*) Arcs from d-tree to the other tree:

$$y_j - y_i + z_{ij} = \tilde{y}_j - \tilde{y}_i + \tilde{z}_{ij}$$

$$= y_j + 13 - y_i + \tilde{z}_{ij} \Rightarrow \tilde{z}_{ij} = z_{ij} - 13$$

so z_{ij} is increased with dual slack of entering arc.

(**) Arcs from the other tree to the J-tree

$$y_j - y_i + z_{ij} = \tilde{y}_j - \tilde{y}_i + \tilde{z}_{ij}$$

$$= y_j - (y_i + 13) + \tilde{z}_{ij}$$

$$\Rightarrow \tilde{z}_{ij} = z_{ij} + 13$$

so z_{ij} is decreased with dual slack of entering arc.

(*) $\check{z}_{dc} = 18 - 13 = 5$

$\check{z}_{ba} = 10 - 13 = -3$

$\check{z}_{ea} = 13 - 13 = 0$

$\check{z}_{fe} = -10 - 13 = -23$

(**) $\tilde{z}_{ad} = -13 + 13 = 0$

$\tilde{z}_{ag} = 1 + 13 = 14$

$\check{z}_{af} = 14 + 13 = 27$