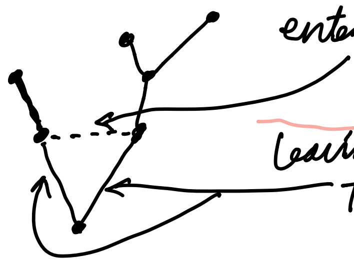
Primal network simplex method

Start with a spanning tree with X_B feasible ($X_B \geq 0$)

1/ Check optimality ($z \geq 0$) (compute y and z)

2/ if not optimal, perform a pivot:



entering variable: some edge (u,v) ($\rightarrow X_{uv}$) enters the spanning tree, creates a cycle

Leaving variable: som edge which leaves the spanning tree (one of those in the cycle we just made)

Applying simplex: X_{uv} can enter if $c_{u,v} > 0$

(u,v) outside the sp. tree — not transformed in simplex.

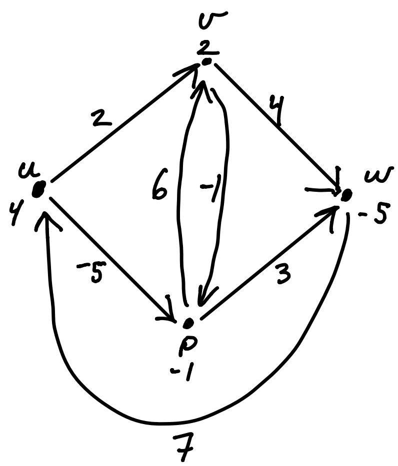
$$\Updownarrow$$

$z_{u,v} < 0$ in the dual solution.

So, find (u,v) outside spanning tree so that $z_{u,v} < 0$, and so that (u,v) introduces a cycle.

Take another edge out of the cycle (we will not compute ratios as we usually do for this)

Example

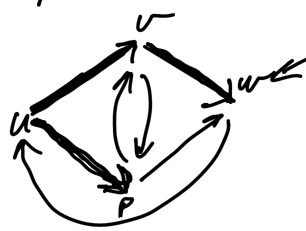


b, c are shown

b = supplies / demands at the nodes

c: costs on the edges

1. Find a spanning tree



w is root node (omit flow balance at w)

2. compute x:

apply flow balance equations at

p: $\quad -x_{up} = -1 \Rightarrow x_{up} = \underline{1}$

u: $\quad x_{up} + x_{uv} = 4 \Rightarrow x_{uv} = \underline{3}$

v: $\quad x_{vw} - x_{uv} = 2 \Rightarrow x_{vw} = x_{uv} + 2 = \underline{5}$

x is feasible

3. Compute y:

$y_j - y_i = c_{ij}$ for $(i,j) \in T$ (T denotes the spanning tree)

(v,w): $\quad 0 - y_v = 4 \Rightarrow \underline{y_v = -4}$

(u,v): $\quad y_v - y_u = 2 \qquad \underline{y_u = -6}$

(u,p): $\quad y_p - y_u = -5 \qquad \underline{y_p = -11}$

4. Compute z (only outside the tree, due to complementary slack

$$y_j - y_i + z_{ij} = c_{ij}$$

$$z_{ij} = c_{ij} + y_i - y_j$$

(w,u): $z_{wu} = c_{wu} + y_w - y_u = 7 + 0 + 6 = \underline{13}$

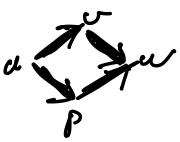
(p,w): $z_{pw} = c_{pw} + y_p - y_w = 3 - 11 - 0 = \underline{-8}$

(p,v): $z_{pv} = c_{pv} + y_p - y_v = 6 - 11 + 4 = \underline{-1}$

(v,p): $z_{vp} = c_{vp} + y_v - y_p = -1 - 4 + 11 = \underline{6}$

Since $z_{pw} < 0$, this is not optimal (it is not dual feasible)

x_{pw} enters basis (maximum coefficient rule)

This creates a loop 

How much can we increase the flow at (p,w)?

Flow balance at nodes in the loop

$$\left\{\begin{array}{lll} p: & x_{pw} = \varepsilon \; : & \tilde{x}_{up} = x_{up} + \varepsilon = 1 + \varepsilon \\ u: & x_{pw} = \varepsilon \; : & \tilde{x}_{uv} = x_{uv} - \varepsilon = 3 - \varepsilon \\ v: & x_{pw} = \varepsilon \; : & \tilde{x}_{vw} = x_{vw} - \varepsilon = 5 - \varepsilon \end{array}\right\}$$

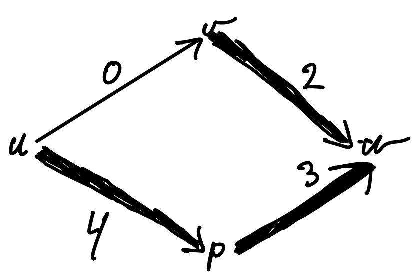
See that we can increase ε to 3, and that x_{uv} then leaves $\Rightarrow x_{uv} = 0$

$x_{pw} = 3$ (entering)

$x_{up} = 4$

$x_{uv} = 0$ (leaving)

$x_{vw} = 2$



Second iteration:

3. compute y: $\quad y_j - y_i = c_{ij}$ for $(i,j) \in T$:

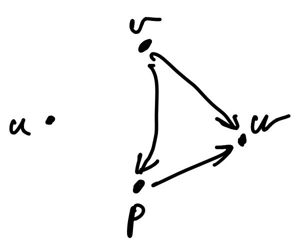
$$(v,w): 0 - y_v = 4 \Rightarrow \underline{y_v = -4}$$
$$(u,p): y_p - y_u = -5 \Rightarrow y_u = y_p + 5 = \underline{2}$$
$$(p,w): 0 - y_p = 3 \Rightarrow \underline{y_p = -3}$$

4. compute z (outside the tree)

$$(w,u): \quad z_{wu} = c_{wu} + y_w - y_u = 7 + 0 - 2 = \underline{5}$$
$$(p,v): \quad z_{pv} = c_{pv} + y_p - y_v = 6 - 3 + 4 = \underline{7}$$
$$(v,p): \quad z_{vp} = c_{vp} + y_v - y_p = -1 - 4 + 3 = \underline{-2}$$
$$(u,v): \quad z_{uv} = c_{uv} + y_u - y_v = 2 + 2 + 4 = \underline{8}$$

Since $z_{vp} < 0$, this is not optimal

X_{vp} enters the basis, giving a cycle in the spanning tree.



Flow balance at w: $\quad X_{vp} = \varepsilon \Rightarrow \tilde{X}_{pw} = X_{pw} + \varepsilon = 3 + \varepsilon$

$\qquad\qquad\qquad\quad v$: $\quad X_{vp} = \varepsilon \Rightarrow \tilde{X}_{vw} = X_{vw} - \varepsilon = 2 - \varepsilon$

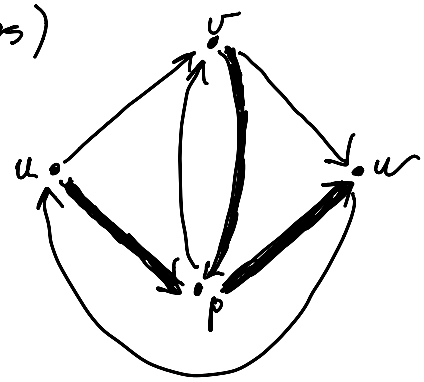
See that we can increase ε to 2, and that X_{vw} leaves

$X_{pw} = \underline{5}$

$X_{vw} = 0$ (leaves)

$X_{up} = \underline{4}$ (did not change, since (u,p) not in loop)

$x_{vp} = \underline{2}$ (enters)



Third iteration

3. Compute y: $\quad y_j - y_i = c_{ij}, \quad (i,j) \in T$:

$$(u,p): \quad y_p - y_u = -5 \Rightarrow \underline{y_u = 2}$$
$$(p,w): \quad 0 - y_p = 3 \Rightarrow \underline{y_p = -3}$$
$$(v,p): \quad y_p - y_v = -1 \Rightarrow \underline{y_v = -2}$$

4. Compute z (outside tree)

$$z_{wu} = c_{wu} + y_w - y_u = 7 + 0 - 2 = \underline{5}$$
$$z_{pv} = c_{pv} + y_p - y_v = 6 - 3 + 2 = \underline{5}$$
$$z_{uv} = c_{uv} + y_u - y_v = 2 + 2 + 2 = \underline{6}$$
$$z_{vw} = c_{vw} + y_v - y_w = 4 - 2 - 0 = \underline{2}$$

This is optimal!

Section 14.4 is on the dual network simplex method.

14.6 Network flows with integer data

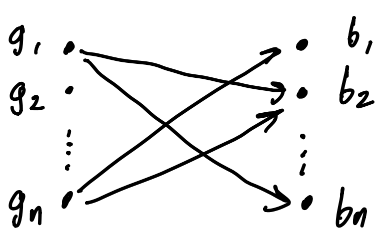
All basic feasible solutions assign integer flows to the arcs.
$\Rightarrow$ optimal solution also has integer flows.

Application of this:

Theorem 14.3 (König's theorem)

We have n girls, n boys
Every boy knows k girls
Every girl knows k boys
Then n marriages can be arranged where everyone knows his/her spouse.

Proof: Represent girls as nodes $g_1, g_2, \ldots, g_n$
—〃— boys —〃— $b_1, b_2, \ldots, b_n$



edges represent that a girl knows a boy.
k edges from each g_i
k edges to each b_j

This is called a bipartite graph.

We will set $b=1$ for all girls (girls are supply nodes)
$b=-1$ for all boys (boys are demand nodes)

This network problem is feasible: set $x_{ij} = \frac{1}{k}$ for all edges:

boy nodes: outflow − inflow $= -\frac{1}{k} - \cdots - \frac{1}{k} = -1$

girl nodes: outflow − inflow $= \frac{1}{k} + \cdots + \frac{1}{k} = 1$

so the flow balance equations are satisfied.

This is an integer valued problem, so we have an integer solution as well.

Since the supplies/demands are ± 1, there can only be one arc with $x=1$ to each (this gives the marriage we seek)

Chap. 15 : Transportation problem
Assignment problem
Shortest path problem